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OEDERED TOPOLOGICAL REPESENTATIONS OF DISTRIBUTIVE LATTICES

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1. Introduction

In 1936, M. H. Stone showed that the category of Boolean algebras with homomorphisms and the category of compact Hausdorff totally disconnected space with continuous maps are dually equivalent. More generally, in [12], he also established that the category $\mathrm{Dist}_{0,1}$ of distributive lattices with zero and unit preserving homomorphisms and the category Spec of spectral spaces with strongly continuous maps are dually equivalent, where a continuous function is strongly continuous if and only if the associated inverse image function maps compact open sets to compact open sets, and a topological space X is a spectral space if it satisfies each of the following properties:

(S1) X is a compact T_0-space

(S2) The set of compact open subsets of X is a ring of subsets of X and a base for the open sets.

(S3) If F is a closed set in X, $\{U_\alpha : \in \Gamma\}$ is a down directed familyof compact open sets of X and $U_\alpha \cap F \neq \phi$, then $\bigcap \{U_\alpha : \alpha \in \Gamma\} \cap F \neq \phi$.

Recently, introducing the concept of a distributive join semilattice, G. Gratzer [3] generalized the previous results.

Let $(X, \leqq)$ ba a partially ordered set and a A a subset of X; then we write

$$d(A) = \{y \in X : y \leqq x \text{ for some } x \in A\},$$
$$i(A) = \{y \in X : x \leqq y \text{ for some } x \in A\}.$$

In particular, if A is a singleton, say $\{x\}$, then we write $d(x)$ (resp. $i(x)$). A subset A of X is said to be decreasing (resp. increasing) if $A = d(A)$ (resp. $A = i(A)$). The order is called discrete if $x \leqq y$ only when $x = y$. By an ordered topological space we mean a triple $(X, T, \leqq)$ such that (X, T) is a topological space and $(X, \leqq)$ is a partially ordered set. When there is no ambiguity we speak of the underlying set X as the ordered topological space. An ordered topological space X is called upper semicontinuous ordered if for each $x \in X$, $i(x)$ is a closed set in X. The space X is said to be totally orderdisconnected

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([7]) if $x \not\leqq y (x, y \in X)$ implies that there is an open increasing set U in X such that $x \in U$ and $y \in X - U$. In [7] and [8], H. A. Priestley showed that the category Todc of totally order-disconnected compact ordered spaces with isotone continuous maps and $\text{Dist}_{0,1}$ are dually equivalent. In [9] and [10], T. P. Speed also established that the category Prof of profinite (i.e. a projective limit of finite discrete topological ordered spaces) with isotone continuous maps and Todc are dually equivalent. In [2], W. H. Cornish developed that Todc and Spec are actually isomorphic categories.

In this paper, we will generalize Gratzer-s result ([3]) and the classical duality of Stone ([12]). For general categorical background and terminology, we refer to [4] and for lattice theory (ordered topological spaces, resp.) to [3], [5, 13], respectively.

2. Ordered Stone Spaces

Let L be a distributive join-semilattice with zero and $S_0(L)$be the Stone space of L (see Grätzer [3]). Define an order relation $\leqq$ on $S_0(L)$ as follows:

$P \leqq Q$ if and only if $Q \in \{\overline{P}\}$, where $\{\overline{P}\}$ denotes the closure of $\{P\}$. Then the order is a partial order on $S_0(L)$, because $S_0(L)$ is T_0-space. It follows that $S_0(L)$ is an ordered topological space, and hence we call $S_0(L)$ the ordered Stone space of L.

REMARKS. (1) It is easy to see that the order $\leqq$ and the set inclusion relation $\subseteqq$ coincide on $S_0(L)$.

(2) For each $x \in L$, $r(x) = \{P \in S_0(L) : x \notin P\}$ is decreasing in $S_0(L)$.

THEOREM 1. *The ordered Stone space $S_0(L)$ of a distributive join semilattice L with zero is characterized (up to iseomorphism) by the following two properties:*

(OS1)$S_0(L)$ is upper semicontinuous ordered in which the compact open decreasing sets form a base for the open sets.

(OS2) If F is a closed increasing set in $S_0(L)$, $\{U_\alpha : \alpha \in \Gamma\}$ is down directed family of compact open decreasing sets of $S_0(L)$, and $U_\alpha \cap F \neq \phi$ for each $\alpha \in \Gamma$, then $(\bigcap_{\alpha \in \Gamma} U_\alpha) \cap F \neq \phi$.

PROOF. Suppose that $(X, T, \leqq)$ is an ordered topological space satisfying the conditions (*OS*1) and (*OS*2). Let L be the set of compact open decreasing sets of X. Then L is a join semilattice with zero under the set inclusion. Let $S_0(L)$ be the ordered Stone space of L and let $P \in S_0(L)$. Then $F = X - U(P)$ is a closed increasing set of X, where $U(P)=\bigcup\{x|x\in P\}$. Let $\{U_\alpha : \alpha\in\Gamma\}$ be the set of compact open decreasing sets of X having the property $U_\alpha\cap F\neq\phi$. It follows that $\{U_\alpha : \alpha\in\Gamma\}=\{x\in L : x\notin P\}$. Hence we have $(\bigcap_{\alpha\in\Gamma} U_\alpha)\cap F\neq\phi$, say $a_p\in(\bigcap_{\alpha\in\Gamma} U_\alpha)\cap F$, by (OS2). It is easy to see that $F=i(a_p)$ and $(\bigcap_{\alpha\in\Gamma} U_\alpha)\cap F =\{a_p\}$. Define a function $f : S_0(L)\longrightarrow X$ by $f(p)=a_p$ for each $P\in S_0(L)$. Firstly we will show that f is an iseomorphism (i.e. an order isomorphism and a topological homeomorphism). To show that f is onto, let a be any point of X and let $I=\{x|x\in L,\ x\subseteq X-i(a)\}$. Then I is an ideal of L and $X-i(a)=U(I)$. Hence I is prime. In fact, let $U,\ V\in L,\ U\notin I,\ V\notin I$. Then $U\subsetneqq X-i(a)$ and $V\subsetneqq X-i(a)$, and so $U\cap i(a)\neq\phi$, $V\cap i(a)\neq\phi$. Since U, V are decreasing in X, we have $a\in U$ and $a\in V$. It follows that $(U\cap V)\subsetneqq X-i(a)$, i.e. $U\cap V\subsetneqq U(I)$. Hence there exists an $W\in L$ such that $W\subseteq U\cap V$ and $W\subsetneqq U(I)$. It follows that $W\notin I$, and hence I is prime. Thus we have $f(I)=a$, i.e. f is onto. To see that f is an order embedding let $P\leqq Q$. Then $U(P)\subseteq U(Q)$ and hence $X-U(P) \supseteq X-U(Q)$. It follows that $i(a_p)\supseteq i(a_q)$. Hence $a_P\leqq a_Q$, i.e. $f(P)\leqq f(Q)$. Similarly we can show that $f(P)\leqq f(Q)$ implies $P\leqq Q$. Since it is easy to see that f is one-to-one, f is an order isomorphism. The left part of the proof is an analogous to the argument given in Grätzer [3, pp. 121—122], we omit it.

COROLLARY 2. *The ordered Stone space of a distributive lattice with zero is characterized by* (OS1), (OS2) *and* (OS3) *The intersection of two compact open decreasing sets is compact.*

COROLLARY 3. *The ordered Stone space of a distributive lattice with zero and unit is characterized by*

(OS1') *The space is compact and upper semicontinuous ordered in which the compact open decreasing sets for a base for the open sets,* (OS2) *and* (OS3).

REMARKS. (1) If the given orders in Theorem 1 and Corollary 2, 3 are discrete, they reduce to Grätzer's result ([3]), Stone theorem([12]), respectively,

(2) Let L be a Boolean algebra. Then the ordered Stone space and the Boolean space of L coincide.

§3 Ordered Stone Duality

Let L be a distributive lattice with 0 and let $\hat{L}$ be the ordered Stone space $S_0(L)$ of L. Let $\hat{\hat{L}}$ be the set of all compact open decreasing sets in $\hat{L}$. Then $\hat{\hat{L}}$ is a distributive lattice with zero under the set inclusion relation.

Define a map $f : L\longrightarrow L$ by $f(a)=\{P\in\hat{\hat{L}} :\ a\notin P\}$ for each $a\in L$. Then f is an isomorphism preserving zero element. Lex X be an ordered topological space with the conditions (OS1), (OS2) and (OS3), and let $\hat{X}$ be the set of all compact open decreasing sets of X : then $\hat{X}$ is a distributive lattice with zero under the set inclusion relation. Lex $\hat{\hat{X}}$ be the ordered Stone space of $\hat{X}$. Then there exists an iseomorphism from $\hat{\hat{X}}$ onto X by Theorem 1.

Denote Dist_0 = the category of distributive lattices with zero and zero preserving homomorphisms.

OSpec = the category of ordered topological spaces satisfying conditions (OS1), (OS2) and (OS3), and isotone, strongly continuous maps.

Define a contravariant functor $F : \mathrm{Dist}_0 \longrightarrow \mathrm{OSpec}$ as follows.

For each $L \in \mathrm{Dist}_0$, let $F(L)=\hat{L}$ and for any homomorphism $f : L \longrightarrow M$ in Dist_0, we define $F(f) : F(M) \longrightarrow F(L)$ by $F(f)(P)=f^{-1}(P)$ for any $P \in F(M)$.

Define a contravariant functor $G : \mathrm{OSpec} \longrightarrow \mathrm{Dist}_0$ as follows.

For each $X \varepsilon O\mathrm{Spec}$, let $G(X)=\hat{X}$, and for any isotone, strongly continuous map $h : X \longrightarrow Y$ in OSpec, we define $G(h) : G(Y) \longrightarrow G(X)$ *by* $G(h)(V)=h^{-1}(V)$ for any $V \in G(Y)$. By a routine verification, we have

THEOREM 4. *Dist_0 and OSpec are dually equivalent.*

REMARK. If given orders of objects of OSpec are discrete, then Theorem 4 reduces to the well-known Stone duality theorem([12]).

Denote Cospec = the category of ordered topological spaces satisfying conditions (OS1'), (OS2) and (OS3), and isotone, strongly continuous maps.

Define a covariant functor $F : \mathit{Spec} \longrightarrow \mathrm{Cospec}$ as follows :

For each $(X, T) \in \mathrm{Spec}$, let $F(X)=(T, T, \leqq)$, where $x \leqq y$ in X iff $y \varepsilon \{\bar{x}\}$, and for any strongly continuous function $f : (X, T) \longrightarrow (Y, U)$ in Spec, we define $F(f) : F(X) \longrightarrow F(Y)$ *by* $F(f)=f$.

Let $G : \mathrm{Cospec} \longrightarrow \mathrm{Spec}$ be the forgetful functor.

By an easy verification, we have

THEOREM 5. *Spec and Cospec are isomorphic.*

COROLLARY 6. *Todc and Cospec are isomorphic.*

COROLLARY 7. *$\mathit{Dist}_{0,1}$ and Cospec are dually equivalent.*

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