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# On the $d$-complement Submodule in the $d$-continuous Module

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## I. Introduction

The notion of dual continuous module is dual to the concept of continuous modules originally developed by Y. Utumi [4]. In 1976 S. Mohamed and T. Bouchy showed that continuous module is a generalization of (quasi-) injective module [2]. But dual continuous module is not generalization of (quasi-) projective modules. For instance, only over a perfect ring, we can have the fact that projective module is dual continuous module [3]. Over an arbitrary ring the relationship between dual continuity and projectiveness is less close.

The purpose of this paper is to establish some properties on dual continuous modules and to show d-complement submodule of dual-continuous module is dual continuous, using Zorn's lemma, and to show some properties on $d$-complement submodule in a dual-continuous module.

## II. Definitions and preliminaries.

We begin with some definitions and some properties which are useful in later our proposition and theorem. Throughout this paper we assume that modules will be right modules over a not necessarily commutative ring with identity element. Referring to modules, sum and summand will mean direct sum and direct summand and a submodule $A$ of $M$ will be denoted by $A \leq M$.

**Definition 2.1.** A submodule $K$ in $M$ is small in $M$, denoted by $K \ll M$, in case for every submodule $L \leq M$ such that $K+L=M$ implies $L=M$.

**Proposition 2.2.** Let $M$ be a module with submodule $K \leq N \leq M$, then $K \ll N \leq M$ implies $K \ll M$. [1].

**Proof.** Let $L \leq M$ such that $K+L=M$, then $N \cap (K+L) = N \cap M = N$, $K+(N \cap L) = N$, $N \cap L = N$, so $N \leq L$, on the other hand $N+L=M$ which implies $L=M$.

**Definition 2.3.** A submodule $L$ in $M$ is called $d$-complement of $N$ in $M$, if $L$ is minimal with $L+N=M$. This is dual concept of $M$-complement in general algebra [1]. Y. MIyashita showed following proposition, we will show again here in some different method.

**Proposition 2.4.** A is $d$-complement of $B$ in $M$ if and only if $A+B=M$ and $A \cap B \ll A$.

**Proof.** Assume $A$ is d-complement of $B$ in $M$. Let $K \leq A$ such that $A \cap B+K=A$, $A \cap B+K+B=A+B=M$. This implies $K+B=M$, since $A$ is a minimal submodule of $M$ such that $A+B=M$, so $K=A$. Conversely, let $K \leq A$ and $K+B=M$, then $A \cap (K+B)=A \cap M=A$, by modular law $K+A \cap B=A$, thus $K=A$, since $A \cap B \ll A$.

Following Y. Utumi [4] we have a definition

**Definition 2.5.** A module $M$ is called continuous, if $M$ satisfies

(I) for every submodule $A$ of $M$ is essential in some direct summand of $M$,

(II) if a module $A$ is isomorphic to some direct summand of $M$, then $A$ is direct summand of $M$.

S. Mohamed and T. Bouchy [2] established some results on continuous modules : for example,

(1) Every quasi-injective module is continuous.

(2) A ring $R$ is semi-simple artinian if and only if every finitely generated $R$-module is continuous.

**Definition 2.6.** A module $M$ is called dual continuous ($d$-continuous) module if $M$ satisfies (I) and (II) :

(I) for every submodule $A$ of $M$, there exists $M=M_1 \oplus M_2$ such that $M_1 \leq A$ and $M_2 \cap A \ll M_2$.

(II) every surjection from $M$ onto arbitrary summand of $M$, splits, *i. e.* Let $A$ be a summand of $M$, for every surjection $f: M \longrightarrow A$, there is a map $g: A \longrightarrow M$ such that $fg=1_A$.

S. Mohamed and B. J. Müller [3] showed that d-continuous module is not generalization of (quasi-) projective module, for instance,

(1) A ring $R$ is perfect if and only if every quasi- projective $R$- module is $d$-continuous.

## Ⅲ. on d-continuous modules

**Proposition** 3. 1. Every free module over a principal ideal domain is a d- continuous module.

**Proof**. Let $F$ be a free module over $P.I.D.$ and $\{e_i\}_{i\in I}$ be a basis for $F$, and let $F'$ be any submodule of $F$ . Then $F'$ has a basis $\{e_j\}_{j\in J}$ where $J\subseteq I$, since any submodule of free module is free over $P.I.D.$ Let $F''$ be the submodule of $F$ generated by $\{e_k\}_{k\in K}$, where $J\cup K=I$ and $J\cap K=\phi$. This $F'$ and $F''$ satisfies the condition (I) of the $d$-continuity.
Now $A$ be any summand of $F$ and let $f: F\longrightarrow A$ be any surjection, we can define $g: A\longrightarrow F$ such $fg=1_A$, using the basis elements.

**Corollary** 3. 2. Every module over a P. I. D. is an epimorphic image of d-continuous module.

**Proof**. Combine the fact that every module is the homomorphic image of a free module and above proposition 3. 1.

**Corollary** 3. 3. For any module $M$ and free module $F$ over P. I. D. If $f: M\longrightarrow F$ and $g: F\longrightarrow M$ be a map such that $gf=1_M$, then $M$ is a $d$-continuous module.

**Proof**. By the assumption $F\cong \mathrm{Ker}g \oplus \mathrm{Im}f$, thus $M\cong F/\mathrm{Ker}g$, so applying this result to the proposition 3. 1. we can get the result.

**Proposition** 3. 4. Every summand of $d$-continuous module is $d$-continuous.

**Proof**. Let $A$ be a summand of $d$-continuous module $M$. For any submodule $A$ in $A$, $A_1$ also submodule of $M$, there exists $M=M_1\oplus M_2$ such that $M_1\leq A_1$ and $M_2\cap A_1\ll M_2$. Now $A$ has a decomposition such that $A=M_1 \oplus (M_2\cap A)$, since $A$ is a summand of $M$. This satisfies the condition (Ⅰ) of the $d$-continuity. Let $A_1$ be a summed of $A$, for any surjection $f_1: A\longrightarrow A_1$, there exists a surjection $f: M\longrightarrow A_1$ such that $f|_A=f_1$. Since there exists $g: A\longrightarrow M$ such that $fg=1_{A_1}$, (Ⅱ) is done.

**Corollary** 3. 5. Let $M$ be a $d$-continuous module and let $e$ be an idempotent in $\mathrm{End}_R(M)$, then $eM$ is a $d$-continuous module.

**Proof**. Since $M=eM\oplus(1-e)M$, $eM$ is a summand, by the proposition 3. 4. $eM$ is a $d$-continuous module.

**Remark**: Converse of proposition 3. 4. is not always true. [3]

**Theorem** 3. 6. Every $d$-complement submodule of $d$-continuous module is $d$-continuous module.

**Proof.** Let $A$ be a $d$-complement submodule of $B$ in $M$. It suffices to show that $A$ is a summand of $M$, since summand of $d$-continuous module is always d-continuous. Consider a family of submodules in $M$ such that $u=\{K\leq M \mid A+K=M$ and $A\cap K\ll A\}$, then $u\neq\phi$, since $B$ is an element in $u$. Now give an ordering on $u$ by the set inclusion reversing. Let $\{S_i \mid i\in I\}$ be any chain in $u$, and let $S=\cap S_i$ for all $i\in I$. then $S$ is an upper bound for the chain $\{S_i \mid i\in I\}$. Thus by the Zorn's lemma, there exists a maximal element $K$ in $u$ Now we claim that $A\cap K=0$. If $A\cap K\neq 0$, there is an element $x(\neq 0)$ in $A\cap K$, take $K'=K-x$, then $K'$ contained in $K$ and $A+K'=M$ moreover $A\cap K'\subsetneq A\cap K\ll A$, by the proposition 2. 2. $A\cap K'\ll A$, this contradicts $K$ is maximal elment in $u$.

**Proposition 3.7.** Let $M$ be a $d$-continuous module. For every pair $(A, B)$ of submodule with $A+B=M$, $A$ contains a $d$-complement of $B$.

**Proof.** Let $B_1$ be a $d$-complement of $B$ in $M$. $B_1=M\cap B_1=(A+B)\cap B_1=A\cap B_1+B\cap B_1$, by the proposition 2.4. $B\cap B_1\ll B_1$, we can get only $A\cap B_1=B_1$. If $A$ is contained in $B_1$ then $A+B\cap B_1=B_1$, thus $A=B_1$, we have the result $B_1$ is contained in $A$.

**Proposition. 3.8.** If $B$ is a summand of the $d$-continuous module $M$, and if $A$ is $d$-complement of $B$ in $M$, then $M=A\oplus B$.

**Proof.** Since $B$ is a summand, we can get $M=B\oplus B'$ for some submodule $B'$ in $M$. Now $A$ is $d$-complement of $B$, $A=M\cap A=(B+B')\cap A=B\cap A+B'\cap A$. since $A\cap B\ll A$, we can have $A\cap B'=A\cdots\cdots$ (1) on the other hand $B'=M\cap B'=(A+B)\cap B'=A\cap B'+B\cap B'$, $B\cap B'=0$ implies $A\cap B'=B'\cdots\cdots$(2). Thus we have $M=A\oplus B$ form (1) and (2).

## References

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