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On Almost-Continuous and δ-Continuous Mappings

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1. Introduction

In 1968 M. K. Singal and A. R. Singal [8] have introduced a weak form of continuity called almost-continuity. In 1980 Takashi Noiri [6] have introduced the concept called δ-continuity. In 1969 M. K. Singal and Ashe Mathur [7] the concept of nearly-compact spaces as a generalization of compact spaces. In 1972, in order to localize the concept of nearly-compact space, D. Carnahan [2] defined N-closed sets analogous to compact sets and also introduced a class of spaces called locally nearly-compact spaces.

The purpose of this note is to study some properties of δ-continuous mappings, almost-continuous mappings and nearly-compact spaces.

Throughout this note spaces always mean topological spaces on which no separation axioms are assumed unless explicitly stated.

A subset A of a space X is said to be ***regular open*** (resp. ***regular closed***), if $\mathring{\bar{A}}=A$ (resp. $\bar{\mathring{A}}=A$), where $\bar{A}$ (resp. $\mathring{A}$) denote the closure (resp. the interior) of A.

A mapping $f: X\to Y$ is said to be ***almost-continuous*** [8], if for each $x\in X$ and each open neighborhood V of $f(x)$, there exists an open neighborhood U of x such that $f(U)\subseteq \mathring{\bar{V}}$.

A mapping $f: X\to Y$ is said to be ***δ-continuous*** [6], if for each $x\in X$ and each open neighborhood V of $f(x)$, there exists an open neighborhood U of x such that $f(\mathring{\bar{U}})\subseteq \mathring{\bar{V}}$. We can see immediatly that if A is open in X, then $A\hookrightarrow X$ is δ-continuous.

2. Some properties of two continuous mappings

A point $p\in X$ is said to be ***δ-cluster point*** of A, if $A\cap U\neq\phi$ for every regular open set U containing p. The set of all δ-cluster points of A are called the δ-

closure of A and denoted by $\bar{A}^\delta$. If $\bar{A}^\delta = A$, A is called *δ-closed*. The complements of a δ-closed set is called *δ-open*. The following properties are easy consequences of the above definition and thus the proofs are omitted.

Proposition 2.1 The following statements are true for any space X:

(1) ϕ and X are both δ-closed subsets.

(2) If $A \subseteq X$, then $\bar{A} \subseteq \bar{A}^\delta$

(3) If $A \subseteq B \subseteq X$, then $\bar{A}^\delta \subseteq \bar{B}^\delta$

(4) If $A \subseteq X$, then $\bar{A}^\delta$ is closed in X.

(5) If A is open in X, then $\bar{A}^\delta = \bar{A}$ and $\bar{\bar{A}}^\delta = \bar{A}$, *i. e.* $\bar{A}$ is δ-closed in X.

(6) Arbitrary intersections and finite unions of δ-closed subsets of X are δ-closed in X.

Proposition 2.2 If $f: X \to Y$ is continuous, then f is almost continuous.

Proof It is clear by $f(U) \subseteq f(\mathring{\bar{U}}) \subseteq \mathring{\bar{V}}$

Proposition 2.3 If $f: X \to Y$ and $g: Y \to Z$ is δ-continuous, then $gof: X \to Z$ is δ-continuous.

Proof Let V be any reguar open set in Z. The $g^{-1}(V)$ is δ-open in Y, since g is δ-continuous. Thus $f^{-1}(g^{-1}(V))$ is δ-open in X, since f is δ-continuous. But $f^{-1}(g^{-1}(V)) = (g \circ f)^{-1}(V)$. Therefore $g \circ f$ is δ-continuous.

Corollary 2.1 If $f: X \to Y$ is δ-continuous and A is any open set in X, then $f \mid A: A \to Y$ is δ-continuous.

Proof Let $i: A \hookrightarrow X$ be the inclusion mapping. Then $f \mid A = f \circ i$ and i is δ-continuous. Therefore, by (5) in proposition 2.1 and proposition 2.3, $f \mid A$ is δ-continuous.

Theorem 2.1 Let $f: X \to Y$ be any mapping. If $\{U_c \mid c \in I\}$ is a regular open covering of X and $f \mid U_c: U_c \to Y$ is δ-continuous, for each $c \in I$, then f is δ-continuous.

Proof Let V be any regular open subset of Y, Then since $f \mid U_c$ is δ-continuous, $(f \mid U_c)^{-1}(V) = f^{-1}(V) \cap U_c$ is δ-open in U_c, for each $c \in I$. Since U_c is a regular open subset of X for each $c \in I$, $f^{-1}(V) \cap U_c$ is δ-open in X. Thus $f^{-1}(V)$ is δ-open in X, and hence f is δ-continuous.

Theorem 2.2 Let $f_c: X_c \to Y_c$ be any mapping for each $c \in I$ and let $f: \prod_{c \in I} X_c \to \prod_{c \in I} Y_c$ be defined by $f((x_c)) = (f_c(x_c))$ for each point $(x_c) \in \prod_{c \in I} X_c$

If f_i is almost-continuous, for each $c \in I$, then f is almost-continuous.

Proof Let $(x_i) \in \prod_{i \in I} X_i$ and let V be any regular open set in $\prod_{i \in I} Y_i$ of $f((x_i))$. Then, there exists an elementary set $\prod_{i \in I} A_i$ of the product topology on $\prod_{i \in I} Y_i$ such that $f((x_i)) \in \prod_{i \in I} \mathring{A}_i \subseteq V$, where all but at most fimitely many $A_i = Y_i$ Since V is regularly-open, $\overline{\prod_{i \in I} A_i} \subseteq V$. Thus, for each L_k, $f_{ik}(x_{ik}) \in A_{ik} \subseteq \mathring{\bar{A}}_{ik}$ and so there exists an open subset U_{ik} of X_{ik} such that $x_{ik} \in U_{ik}$ and $f_{ik}(x_{ik}) \in f_{ik}(U_{ik}) \subseteq \mathring{\bar{A}}_{ik}$. Thus $\prod_{i \in I} U_i$ is an open set containing (x_i) such that $f(\prod_{i \in I} U_i) \subseteq V$, where $U_i = X_{i}$ $_{i_k \neq i}$ h, for $k = 1, 2, \cdots n$.
Therefore f is almost-continuous.

3. Nearly-compact spaces

Since the concept of nearly-compact spaces was introduced by M. K. Singal and A sha Mathur, its properties has been studied by James E. Joseph, Travis Thompson, Larry L. Herrington and Takashi Noiri.

A mapping $f: X \to Y$ is said to be θ-continuous, if for each $x \in X$ and each open neighborhood V of $f(x)$, there exists an open neighborhood U of x such that $f(\bar{U}) \subseteq \bar{V}$.

A mapping $f: X \to Y$ is said to be ***almost-open*** [8], if for each regular open U of X, $f(U)$ is open in Y. It is clear that every open mapping is almost-open.

Detinition 3. 1 (i) A subset A of a space X is said to be ***N-closed relative*** to X [2], if every covering of A by regular open sets in X has a finite subcover.

(ii) A space X is said to be ***nearly-compact*** (NC) [7], if X is N-closed relative to X.

Theorem 3. 1 If $f: X \to Y$ is a δ-continuous surjection and X is NC, then Y is NC.

Proof If $f: X \to Y$ is δ-continuous and A is N-closed relative to X, then $f(A)$ is N-closed relative to Y [6].
It follows from (ii) in Definition 3. 1

Proof of Theorem 3. 2 Since every almost-continuous mapping is θ- continuous [5], it is sufficient to show that f is δ-continuous. Let $x \in X$ and let V be any open neighborhood of f(x) Then since f is θ-continuous, there exists an open neighborhood U of x suck that $f(\bar{U}) \subseteq \bar{V}$. Since $\mathring{\bar{U}} \subseteq \bar{U}$, $f(\mathring{\bar{U}}) \subseteq \bar{V}$. Since f is almost-open, $f(\mathring{\bar{U}}) \subseteq \mathring{\bar{V}}$. Therefore, by Theorem 3. 1, $Y = f(x)$ is NC.

References

〔1〕 N. Bourbaki, ***Topologie Générale***, 3*rd. ed.* Actualites Sci. Indust. no. 1142, Hermann, Paris 1961.

〔2〕 D. Carnahan, ***Locally nearly-compact spaces***, Boll., Un. Math. Ital. (4) 6 (1972), 146-153.

〔3〕 L. L. Herrington, ***Properties of nearly-compact spaces***, Proc. Amer. Math. Soc. 45 (1974), 431-436.

〔4〕 J. E. Joseph, ***Characterization of nearly-compact spaces***, Boll. Un. Math. Ital. (5) 13-B (1976), 311-321.

〔5〕 Hong Oh Kim, ***Notes on c-compact spaces and functionally compact spaces***, Kyungpook Math. J. 10 (1970), 75-80.

〔6〕 T. Noiri, ***On δ-continuous functions***, J. Korean. Math. Soc. Vol. 16, No. 2, 1980. 161-166.

〔7〕 M. K. Singal and Asha Mathur, ***On nearly-compact spaces***, Boll. Un. Math. Ital. (4) 2 (1969), 702-710.

〔8〕 M. K. Singal and A. R. Singal, ***Almost-continuous mappings***, Yokohama Math. J. 16 (1968), 63-73.

〔9〕 T. Thompson, ***Characterizations of nearly-compact spaces***, Kyungpook Math. J. 17 (1977), 37-41.

〔10〕 N. V. Velicko, ***H-closed topological spaces***, Amer. Math. Soc. Transl. (2) 78 (1968) 103-118.

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