HONAM MATHEMATICAL JOURNAL
Volume 5, Number 1, September, 1983

On Strongly δ-Continuous Mappings

By Kul Hur

1. Introduction

The purpose of this note is to introduce a new class of mappings called strongly δ-continuous mapping of strong form of continuous mapping and to investigate some properties of this mapping.

Throughout this note spaces always mean topological spaces on which no separation axioms are assumed unless explicitly stated.

A subset of a space X is said to be ***regular open*** (resp. ***regular closed***), if $\overset{\circ}{\overline{A}} = A$ (resp. $\overline{\overset{\circ}{A}} = A$), where $\overline{A}$ (resp. $\overset{\circ}{A}$) denote the closure (resp. the interior) of A. A point $p \in X$ is said to be ***δ-cluster point*** of A, if $A \cap U \neq \phi$ for every regular open set U containing p. The set of all δ-cluster points of A are called the ***δ-closure*** of A and denoted by $\overline{A}^{\delta}$. If $\overline{A}^{\delta} = A$, A is called ***δ-closed.*** The complement of δ-closed is called ***δ-open.*** The following properties are easy consequences of the above definition and thus are omitted.

Proposition 1.1 The following statements are true for any space X:

(1) ϕ and X are both δ-closed subsets.

(2) If $A \subseteq X$, then $\overline{A} \subseteq \overline{A}^{\delta}$

(3) If $A \subseteq B \subseteq X$, then $\overline{A}^{\delta} \subseteq \overline{B}^{\delta}$

(4) If $A \subseteq X$, then $\overline{A}^{\delta}$ is closed in X.

(5) If A is open in X, then $\overline{A}^{\delta} = \overline{A}$ and $\overline{\overline{A}}^{\delta} = \overline{A}$, *i. e.*, $\overline{A}$ is closed in X.

(6) Arbitrary intersections and finite unions of δ-closed subsets of X are δ-closed in X.

2. Characterizations and basic properties

Definition 2.1 A mapping $f: X \to Y$ is said to be ***strongly δ-contiunous,*** if for each $x \in X$ and each open neighborhood V of $f(x)$, there exists a regular open neighborhood U of x such that $f(\overset{\circ}{\overline{U}}) \subseteq V$.

Theorem 2. 1 (Characterization)

For any mapping $f: X \to Y$, the following are equivalent :

(1) f is strongly δ-continuous.

(2) For each $x \in X$ and each open set V containing $f(x)$, there exists a regular open set U containing x such that $f(U) \subseteq V$.

(3) $f(\overline{A}^{\delta}) \subseteq \overline{f(A)}$, for every $A \subseteq X$.

(4) $\overline{f^{-1}(B)}^{\delta} \subseteq f^{-1}(\overline{B})$, for every $B \subseteq X$.

(5) For every closed subset F of Y, $f^{-1}(F)$ is δ-closed in X.

(6) For every open subset V of Y, $f^{-1}(V)$ is δ-open in X.

Proof (1)⇒(2) : It is clear from Definition 2. 1 (2)⇒(3) : Suppose that $f(x) \notin \overline{f(A)}$

Then there exists an open neighborhood V of $f(x)$ such that $V \cap f(A) = \phi$.

Thus there exists a regular open set U of x such that $f(U) \subseteq V$ and so $x \in U \subseteq f^{-1}(V)$.

But $U \cap A \subseteq f^{-1}(V) \cap A \subseteq f^{-1}(V) \cap f^{-1}(f(A)) = f^{-1}[V \cap f(A)] = \phi$.

Thus $x \notin \overline{A}^{\delta}$ and so $f(x) \notin f(\overline{A}^{\delta})$.

Therefore $f(\overline{A}^{\delta}) \subseteq \overline{f(A)}$.

(3)⇒(4) : Suppose that $x \notin f^{-1}(\overline{B})$. Then $f(x) \notin \overline{B}$.

But $f[\overline{f^{-1}(B)}^{\delta}] \subseteq \overline{f[f^{-1}(B)]} \subseteq \overline{B}$.

Thus $f(x) \notin f[\overline{f^{-1}(B)}^{\delta}]$ and so $x \notin \overline{f^{-1}(B)}$

Therefore $\overline{f^{-1}(B)}^{\delta} \subseteq f^{-1}(\overline{B})$.

(4)⇒(5) & (5)⇒(6) : It is clear.

(6)⇒(1) : Let $x \in X$ and let V be any open neighborhood of $f(x)$.

Then $f^{-1}(V)$ is δ-open in X and $x \in f^{-1}(V)$.

Thus $Cf^{-1}(V)$ is δ-closed in X and $x \notin Cf^{-1}(V)$

Thus there exists a regular open set U of x such that $U \cap Cf^{-1}(V) = \phi$, and so $U \subseteq f^{-1}(V)$, *i. e.*, $f(U) \subseteq V$.

Therefore f is strongly δ-continuous.

Theorem 2. 2 If $f: X \to Y$ is strongly δ-continuous, then f is continuous.

Proof Let F be any closed subset of Y. Then by (5) in Theorem 2. 1, $f^{-1}(F)$ is δ-closed in X, *i. e.*, $\overline{f^{-1}(F)}^{\delta} = f^{-1}(F)$. Therefore, by (4) in proposition 1. 1, $f^{-1}(F)$ is closed in X.

However the converse of Theorem 2. 2 does not hold. Consider the following example :

Example 2.1 Let $X=Y=\{a, b, c\}$, $\mathcal{J}_Y=\{\phi, \{a\}, \{c\}, \{a, c\}, X\}$ and $\mathcal{J}_X=\{\phi, \{a\}, \{c\}, \{a, b\}, \{a, c\}, X\}$. Let $f: (X, \mathcal{J}_X) \to (Y, \mathcal{J}_Y)$ be the identity mapping. Then f is continuous but not strongly δ-continuous.

The family of regular open subsets of X forms a base of a topology on the underlying set of a space X[1].
This new space is called the ***semi-regularization*** of X and is denoted by X^* in this note. If $X^*=X$, then X is said to be ***semi-regular***. A set is δ-closed in X if and only if it is closed in X^*.

Theorem 2.3 If $f: X \to Y$ is continuous and X is semi-regular, then f is strongly δ- continuous.

Proof Let $x \in X$ and let V be any open neighborhood of $f(x)$. Then there exists an open neighborhood U of x such that $f(U) \subseteq V$. Since X is semi-regular, there exists a regular open U_0 such that $U_0 \subseteq U$. Thus $f(U_0) \subseteq V$. Therefore f is strongly δ-continuous.

We define a mapping $f^*: X^* \to Y$ associated with a mapping $f: X \to Y$ as follows : $f^*(x)=f(x)$ for each $x \in X^*$.

Corollary 2.1 A mapping $f: X \to Y$ is strongly δ-continuous if and only if $f^*: X^* \to Y$ is continuous.

Proof It follows from Theorem 2.2 and 2.3.

Theorem 2.4 If $f: X \to Y$ and $g: y \to Z$ are strongly δ-continuous, then $gof: X \to z$ is strongly δ-continuous.

Proof Let $x \in X$ and each open set V containing $(gof)(x)=g(f(x))$. Then there exists an open set U' of $f(x)$ such that $g(U') \subseteq V$. Thus there exists a regular open set U of x such that $f(U) \subseteq U'$, and hence there exists a regular open set U of x such that $(gof)(U) \subseteq V$. Therefore gof is strongly δ-continuous.

Definition 2.2 A mapping $f: X \to Y$ is said to be ***almost-continuous*** [5] (resp. ***δ-continuous*** [3], ***θ-continuous*** [2], ***strongly θ-continuous***), if for each $x \in X$ and each open neighborhood V of $f(x)$, there exists an open neighborhood U of x such that $f(U) \subseteq \mathring{\bar{V}}$ (resp. $f(\mathring{\bar{U}}) \subseteq \mathring{\bar{V}}$, $f(\bar{U}) \subseteq \bar{V}$, $f(\bar{U}) \subseteq V$).

The following statements are easy consequences and thus the proofs are omitted :

Theorem 2.5 Let $f: X \to Y$ be any mapping.

(1) If f is strongly δ-continuous, then f is δ-continuous

(2) If f is strongly θ-continuous, then f is strongly δ-continuous.

Definition 2.3 A space X is said to be ***almost-regular*** [4], if for each regular closed set $F\subseteq X$ and each $x\notin F$, there exists disjoint open sets U and V in X such that $x\in U$ and $F\subseteq V$.

Theorem 2.6 If $f: X\to Y$ is strongly δ-continuous, and X is almost- regular, then f is strongly θ-continuous.

Proof Let $x\in X$ and let V be any open neighborhood of $f(x)$. Then there exists an open neighborhood U of x such that $f(\mathring{\bar{U}})\subseteq V$. Since X is almost- regular, there exists a regular open U_0 such that $x\in U_0\subseteq \bar{U}_0\subseteq \bar{U}$ [4]. Thus $f(\bar{U}_0)\subseteq f(\mathring{\bar{U}})\subseteq V$. Therefore f is strongly θ-continuous.

References

[1] N. Bourbaki, ***Topologie Generale***, 3rd ed. Actualites Sci. Indust. no. 1142. Hermann, Paris, 1961.

[2] S. Formin, ***Extensions of topological spaces***, Ann. of Math. 44 (1943), 471-480.

[3] T. Noiri, ***On δ-continuous functions***, J. Korean Math. Soc. Vol. 16, No. 2, 1980, 161-166.

[4] M. K. Singal and S. P. Arya, ***On almost-regular spaces***, Glasnik Math. 4 (24) (1969), 89-99.

[5] M. K. Singal and A. R. Singal. ***Almost-continuous mappings***, Yokohama Math. J. 16 (1968), 63-73.

Wonkwang University