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The Structure of Convergence on Function Spaces

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1. Introduction

The study on the function space topologies has mainly been investigated in the space of continuous function (4).

Recently, Skorokhod (1) defined new topologies on the space of all discontinuous of the first kind connection with a problem in probability theory.

In this paper, we investigate the important properties of the continuous convergence structure and Skorokhod convergence structure.

And we show that among these Skorokhod convergent topologies, J_2-convergent topology coincides with the graph topology (Theorem 4.3), almost convergent topology is coarser than M_2-convergent topology (Theorem 4.9), J_1-convergent topology coincides with the structure of continuous convergence (Proposition 3.5), and the uniform convergence topology is contained in the graph topology (Theorem 4.4).

2. The structure of Skorokhod convergence

Definition 2.1: Let (X, d) and (Y, d^*) be completely metric separable spaces. We define the metric R by

$$R((x_1, y_1), (x_2, y_2)) = d(x_1, x_2) + d^*(y_1, y_2).$$

Then we obtain skorokhod convergent topological space (Y^x, R).

Definition 2.2: We denote by $K(X, Y)$ the space of all functions which are defined on the interval $X=[0, 1]$, whose values lie in Y, and which at every point have a limit on the left and continuous on the right (and on the left at $t=1$).

Let us consider certain properties of the functions which belong to $K(X, Y)$. A function $f(x)$ will be said to have a discontinuity

$$d^*(f(x_0-0), f(x_0+0)$$

at the point x_0.

Lemma 2.3 : If $f(x) \in K(X, Y)$, then for any positive ε there exists only a finite number of value of x such that the discotinuity of $f(x)$ is greater than ε.

Proof : This follows from the fact that if there exists a sequence for which $x_k \cdots x_0$ such that

$$d^*(f(x_k+0),\ f(x_k-0)) > \varepsilon,$$

then at x_0 the function $f(x)$ would have no limit either on the right or on the lift.

Lemma 2.4 : Let $x_1, x_2, x_3, \cdots x_k$ be all the points at which $f(x)$ has discontinuities no less than ε. Then there exists *a* δ such that if $(x'-x'') < \delta$ and if x' and x'' both being to the same one of the intervals $(0, x_1)$, $(x_1, x_2), \cdots (x_2, x_3)\ (x_k, 1)$, then $d^*(f(x'),\ f(x'')) < \varepsilon$,

Proof : Assume the contrary. Then there would exist sequenes x_n' and x_n'' which converge to some point x_0 and belong to the same one of the intervals $(0, x_1) \cdots (x_k, 1)$ and the sequences would have the property that

$$d^*(f(x_n'),\ f(x_n'')) \geq \varepsilon.$$

Now the points x_n' and x_n'' lie on opposite sides of x_0 (otherwise $d^*(f(x_n'), f(x_n''))\geq \varepsilon$ would be impossible), so that $d^*(f(x_0+0),\ f(x_0-0)) \geqq \varepsilon$. Therefore x_0 is one of the points $x_1, x_2, \cdots x_k$, which contradicts the statement that x_n' and x_n'' belong to the same one of the intervals $(0, x_1)$, $(x_1, x_2), \cdots (x_k, 1)$.

Definition 2.5 : The sequence of functions $f_n(x)$ converges uniformly to $f(x)$ at the point x_0 if for all $\varepsilon > 0$ there exists a δ such that

$$\lim_{n\to\infty} \sup_{|x-x_0|<\delta} d^*(f_n(x),\ f(x_0)) < \varepsilon.$$

Obviously if $f_n(x)$ converges uniformly to $f(x)$ at every point of some closed set, then $f_n(x)$ converges uniformly to $f(x)$ on this wholeset.

Definition 2.6 : The sequence $f_n(x)$ is called J_1-convergent to $f(x)$ if there exist a sequence of continuous one-to-one mappings $\lambda_n(S)$ of the interval $X=[0, 1]$ onto itself, such that

$$\lim_{n\to\infty} \sup_{s} R((s, f_n(s)),\ (\lambda_n(s),\ f(\lambda_n(s))) = 0.$$

The uniform convergent topology U and J_1-convergent topology J_1 take the form of a single jump at a discontinuity point x_0. In both these topologies for values of x close to x_0, the function $f_n(x)$ can take on values which are either close to

$f(x_0-0)$ or to $f(x_0+0)$.

If we wish to keep this last property, but do not require that the transition be in the form of a single jump, that a function $f_n(x)$ may change back and forth between the values $f(x_0-0)$ and $f(x_0+0)$ several times in the neighborhood of a point x_0, then we obtain topology J_2.

Definition 2.7: A sequence $f_n(x)$ is said to be J_2-convergent to $f(x)$ if there exists a sequence of one-to-one mapping $\lambda_n(s)$ of the interval $X=[0,1]$ onto itself such that

$$\lim_{n\to\infty} \sup_{s} R((s, f_n(s)), (\lambda_n(s), f(\lambda_n(s))) \to 0.$$

Definition 2.8: The pair of functions $(x(s), f(s))$ gives a parametric representaion of the graph (x, y) if those and only those pairs (x, y) belong to it for which an s can be found such that $y=f(s)$, where $f(s)$ is continuous, and $x(s)$ is continuous and monotonically increasing (the functions $f(s)$ and $x(s)$ are defined on the segment $[0,1]$). We note that if $(f_1(s), x_1(s))$ and $(f_2(s), x_2(s))$ are parametric representations of $x(s)$, there exists a monotonically increasing function $\lambda(s)$ such that

$$f_1(s)=f_2(\lambda(s)) \quad \text{and} \quad x_1(s)=x_2(\lambda(s)).$$

Definition 2.9: The sequence $f_n(x)$ is called M_1-convergent to $f(x)$ if there exist parametric representations $(x(s), f(s))$ of $f(x)$ and $(x_n(s), f_n(s))$ of $f_n(x)$ such that

$$\lim_{n\to\infty} \sup_{s} R((x_n(s), f_n(s)), (x(s), f(s)))=0,$$

We can characterize the topology M_1 in the following way from the point of view of the behavior at a point of discontinuity x_0 of the function $f(x)$.

The transition $f(x_0-0)$ to $f(x_0+0)$ is such that first $f_n(x)$ is arbitrarily close to the segment $[f(x_0-0), f(x_0)]$ and second that $f_n(x)$ moves from $f(x_0-0)$ to $f(x_0)$ always advancing.

Definition 2.10: The sequence $f_n(x)$ is called M_2-convergent to $f(x)$ if

$$\lim_{n\to\infty} \sup_{(x_1, y_1)\in G(f(x))} \inf_{(x_2, y_2)\in G(f_n(x))} R(x_1, y_1), (x_2, y_2))=0,$$

Let G be any of our topologies. We shall denote convergence in the topology G by the symbol

$$f_n(x) \xrightarrow{\quad G \quad} f(x).$$

Let us consider the relation between our topologies. It is clear that U is stronger than J_1, and that this in turn is stronger that J_2. It is also clear that M_1 is stronger than M_2. We recall that a topology G_1 is stronger than G_2 if convergence in G_1 implies convergence in G_2. If X is a linear space, we can use any of our topologies. It is easily seen that convergence in M_2 follows from convergence in any of the other topologies, and that convergence in J_1 implies convergence in any of other topologies except ordinary uniform convergent topology U. Denoting the statement "the topology G_1 is stronger than G_2" by $G_1 \rightarrow G_2$, all the above can be summarized by

$$\begin{array}{ccccc} & & M_1 & & \\ & \nearrow & & \searrow & \\ U \longrightarrow J_1 & & & & M_2 \\ & \searrow & & \nearrow & \\ & & J_2 & & \end{array}$$

Example :

Let X be the real line. We set

$$x(t) = \begin{cases} 0 : t < \frac{1}{2}, \\ 1 : t \geqq \frac{1}{2}, \end{cases}$$

$$x'_n(t) = \begin{cases} 0 : t < \frac{1}{2} - 1/n. \\ n(t - \frac{1}{2}) + \frac{1}{2} : \frac{1}{2} - 1/n \leqq t < \frac{1}{2} + 1/n, \\ 1 : t > \frac{1}{2} + 1/n, \end{cases}$$

$$x''_n(t) = \begin{cases} 0 : t < \frac{1}{2} - 1/n, \\ 1 : \frac{1}{2} - 1/n \leqq t < \frac{1}{2}, \\ 0 : \frac{1}{2} \leqq t < \frac{1}{2} + 1/n, \\ 1 : t \geqq \frac{1}{2} + 1/n, \end{cases}$$

Examples can be used to show that the topologies M_1 and J_2 cannot be compared. That is, $x'_n(t) \xrightarrow{M_1} x(t)$ although this does not converge to $x(t)$ in J_2, while $x''_n(t) \xrightarrow{J_2} x(t)$ although this does not converge to $x(t)$ in M_1).

3. The structure of continuous convergence

The structure of continuous convergence Λ will be defined on the function space $C(X, Y)$, the set of all continuous maps from a convergence space X into a convergence space Y.

Definition 3-1 : The structure of continuous convergence will be a convergence structure closely connected with the evaluation map.

$$w : C(X, Y) \times X \rightarrow Y,$$

which sends each couple (f, p) into $f(p)$. In fact, a filter θ belongs to $\Lambda(f)$ if

$$\omega(\theta \times \phi) \rightarrow f(p) \in Y$$

for any filter ϕ converging to $p \in X$ and any point $p \in X$. The symbol $\theta \times \phi$ denotes the filter generated by all sets of the form $T \times F$, where $T \in \theta$ and $F \in \phi$. The set $C(X, Y)$ equipped with Λ is denoted by $C_c(X, Y)$.

A convergence structure Λ on $C(X, Y)$ is called w-admissible if

$$w : (C(X, Y), \Lambda) \times X \longrightarrow Y$$

is continuous.

Definition 3. 2 : A suitable class of convergence spaces, in which any two objects are homeomorphic iff their function algebras endowed with the continuous convergence structure are continuously isomorphic, is called c-embedded spaces.

Proposition 3. 3 : Let X be a c-embedded convergence space. For any ideal $J \subset C(X)$ let $N_x(J)$ be the collection of all points at which every function in J vanishes. An ideal $J \subset C_1(X)$, the algebra $C(X)$ endowed with 1, is closed iff $J = 1(N_x(J))$, the set of all functions in $C(X)$ vanishing on $N_x(J)$. Hence a maximal ideal is closed iff it consists of all functions in $C(X)$ vanishing on a single point. (2)

Definition 3. 4 : The initial convergence structure on $C(X)$ is a locally convex topology T. we call T the locally convex topology associated to Λ.

Proposition 3. 5 : For any c-embedded convergence space X the locally convex topology on $C(X)$ associted with I is the topology of compact convergence. (3)

Corollary 3. 6 : For any c-embedded convergence space X the locally convex topology on $C(X)$ associated with continuous convergence structure is the topology of compact convergence.

4. Comparison of topologies on function spaces

Definition 4. 1 : $f \in Y^x$ is said to be almost continuous in $(X, L) \times (Y, S)$ iff for each open set $U \in L \times S$ containing

$$G(f) = \{(x, f(x)) : x \in . X\} \subset X \times Y$$

there exists a $g \in C(L, S)$ such that $G(g) \subset U$.

(1) $A(L, S)$ will denote the set of almost contions functions in $L \times S$

(2) For each $U \in L \times S$, let $G_u = \{ f \in Y^x : G(f) \subset U \}$.
The topology G on Y^x generated by $\{G_u : U \in L \times S\}$ is called graph topology.

Proposition 4.2 : Let X and Y be completely metric separable. Then the topology of compact covergence K coincides with the graph topology G. (5)

Theorem 4.3 : Let X and Y be completely metric separable. The graph topology coincides with the J_1-convergent topology.

Proof : Let

$$G_u = \{f \in Y^x : G(f) \subset W \in L \times S\}.$$

Then G_u is a subbasic open set in graph topology. we define

$$W = (X - x_0) \times Y \cup X \times S_{d^*}(y_0, \varepsilon) \text{ as } \varepsilon \to 0,$$

and $$S_R(f, \varepsilon) = \{g \in Y^x \mid R((x_0, y \quad (x, g(x)) < \varepsilon\}.$$

Since the metric, in J-convergent topology,

$$\lim_{n\to\infty} \sup R((s, f(s)), (\lambda_n(s), f(\lambda_n(s)))) = 0,$$

if $\lambda_n(s) = x_n, y_0 = f(s), s = x$ and $y_n = f(\lambda_n(s))$, then

$$\lim_{n\to\infty} \sup d^*(y, y_n) = 0$$

and

$$\lim_{n\to\infty} \sup d(x, x_n) = 0.$$

Therefore $G_u = S_R(f, \varepsilon)$ where $S_R(f, \varepsilon)$ is a subbasic open set in J_1-convergent topology.

Theorem 4.4 : Let X and Y be uniform spaces with uniformities μ and Υ respectively. Let $F' \subset F$ consist of functions which are uniformly continuous relative to μ and γ. Then the *u. c.* topology for F' is contained in the graph topology.

Proof : A basis for the u. c. uniformity for F' consists of sets of the form $W(V) = \{(f, g) \in F' \times F' \mid (f(x), g(x)) \in V \in \nu$ for all $x \in X\}$. Consider $W(V)(f)$ where $f \in F'$, $V \in \gamma$. Let $V_2 \in \gamma$ be such that $V_2 \circ V_2 \subset V$. Since $f \in F'$, corresponding to V_2 there exists a $V_1 \in \mu$ such that for all $p, q \in X$, $(p, q) \in V_1$ implies $(f(p), f(q)) \in V_2$. We may, without any loss of generalilty, suppose that V_1, V_2 are symmetric.
Then $U = \cup_{x \in X}\{V_1(x) \times V_2(f(x))\}$ is an open set in $X \times Y$ containing $G(f)$.
Let $g \in F'_u = F' \cap F_u$ *i. e.* $G(g) \subset U$.
For an arbitrary $p \in X$, there exists a $q \in X$ such that $(p, g(p)) \in V_1(q) \times \dot{V}_2(f($ $))$. But then $f(q) \in V_2(f(P))$ and so $g(p) \in V_2 \circ V_2 (f(p)) \subset V(f(p))$. This shows

that $F'_u \subset W(V)(f)$ *i.e.* $W(V)(f)$ is open in (F', G).

Definition 4.5 : Let (X, L) and (Y, S) be topological space. For each pair of open sets $U \in L$ and $V \in S$, let

$$A(U, V) = \{f \in Y^x : f(U) \cap V \neq \phi\}.$$

Almost convergent topology on Y^x is the topology which has as a subbasic $A(U, V)$.

Theorem 4.6 : Almost convergent topology is coarser than point open topology.

Proof : Let $A(U, V)$ be a subbasic open *n.b.h.d.* in almost convergent topology and $f \in A(U, V)$ where $U \in L$, $V \in S$ and L, S are topologies in the domain and range space respectively.

Then there exist $x \in U$ such that $f(x) \cap V \neq \phi$ which implies $f(x) \in V$ and $f \in P(x, V)$.

Therefore, $P(x, V)$ is an open *n.b.h.d.* of f and contained in $A(U, V)$.

Corollary 4.7 : Almost convergent topology is strictly smaller than the point open topology except that they coincide when $(X, L,)$ is the discrete topology.

Proof : If (X, L) is a discrete space, then

$$P(x, V) = \{f \in Y^x : f(x) \in V\}$$

and $x \in L$ which implies $P(x, V) = A(x, V)$.

Theorem 4.8 : Almost convergent topology is coarser than M_2-convergent topology.

Proof : Let (X, L) and (Y, S) be topological space and we difine

$$A(U, V) = \{f \in Y^x : f(U) \cap V \neq \phi\}$$

where $U \subset L$ and $V \in S$,

If A is generated by $A(U, V)$, (Y^x, A) is said to be almost covergent topology.

Since M_2 — metric is

$$R((x_1, y_1), (x_2, y_2)) = d(x_1, x_2) + d^*(y_1, y_2)$$

If we take

$$\varepsilon_0 = \tfrac{1}{2} d^*(y_1, y_2), \quad \delta_0 = \tfrac{1}{2} d(x_1, x_2), \quad x_0 \text{ and } y_0$$

such that

$$x_0 = \inf \{x \mid d(x_1, x) = d(x, x_2)\}$$

and

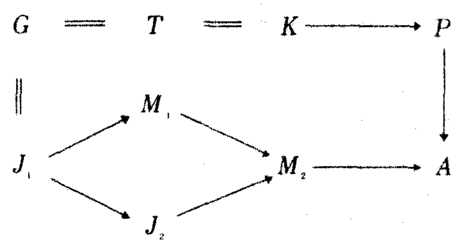
$$y_0 = \inf \{y \mid d^*(y_1, y) = d^*(y, Y_2)\},$$

we can express

$$U = S_d(x_0, \delta_0) = \{z : d(x_0, z) < \delta_0\} \text{ and } V = S_d^*(V_0, \varepsilon_0)$$
$$= \{z : d^*(y_0, z) < \varepsilon_0\}.$$

And so, we can make an open sphere $S_R(f, \varepsilon) = \{g \in Y^x \mid R((x_0, y_0), (x, g(x))) < \varepsilon\}$ where $\varepsilon = \min\{\delta_0, \varepsilon_0\}$, $x \in U$ and $g(x) \in V$. then $A(U, V)$ contains $S_R(f, \varepsilon)$.

Corollary 4.9 : Let X and Y be completely metric separable. Then by (4.3), (4.8), (3.5), and (4.4), we have the following relation :



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