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On the H^P Space with $0<P<1$

By Hyun-Joon Choi

1. Introduction

The space H^p is defined to be the class of all functions $f\equiv f(z)$ which are regular on the interior of the unit circle and such that $\underset{0\leq r<1}{Sup}\int_0^{2\pi}|f(re^{i\theta})|^p d\theta<\infty$.

For arbitrary $f\in H^p$ we define "norm of f" as

$\|f\| = \underset{0\leq r<1}{Sup}((1/2\pi)\int_0^{2\pi}|f(re^{i\theta})|^p d\theta)^{1/p}$.

When $0<p<1$, H^p becomes a complete, perfectly seperable, linear topological space under the topology : $U\subseteq H^p$ is open in case for arbitrary $f_0\in U$ it is true that there exists $r>0$ such that $E_f[\|f-f_0\|<r]$ (sphere of radius r about f_0) lies in U. In fact H^p is linearly homeomorphic to a closed subspace of $L^p[0,2\pi]$. We will show that properties in this paper.

And, we will investigate some important properties of linear functionals in H^p space with $0<p<1$, finally we will show that a weakly convergent sequence in H^p converges uniformly on any compact subset of the unit circle to its weak limit.

2. Preliminaries

The following inequalities, valid for $0<p<1$, are easily established by a consideration of the function $(1+t^p)/(1+t)^p$

(1) $(a+b)^p\leq a^p+b^p$, $a\geqq 0$, $b\geqq 0$.

(2) $(a^p+b^p)^{1/p}\leqq 2^{(1-p)/p}(a+b)$, $a\geqq 0$, $b\geqq 0$.

However, Theorem 19. p. 19 of [3] yields (1), while Theorem 13. p. 24 of the same reference yields (2) by letting k and b of that reference be respectively p and unity.

From the definition of the norm in H^p, $0<p<1$, and (1) it is clear that f and g are in H^p we have

(3) $\|f+g\|^p\leqq\|f\|^p+\|g\|^p$

Inequality (2) yields $(\|f\|^p+\|g\|^p)^{1/p}\leqq 2^{(1-p)/p}(\|f\|+\|g\|)$, whence, we obtain from (3)

(4) $\| f+g \| \leqq 2^{(1-p)/p} (\| f \| + \| g \|)$

3. The topology of H^p

Theorem 3. 1 H^p is a linear topological space.

Proof. That H^p is a vecter space is implied by either of the inequalities(3)or (4) and the obvious homogenity of the norm. It is clear that the union of any number of open sets is open and the intersection of any two open sets is open. We note that the spheres in H^p are open sets as a direct consequence of (3).

As a result of this, inequlity (3) or (4), and the fact that if the norm of an element of H^p vanishes then the element is necessarily the zero element of H^p, we conclude that the Hausdorff seperation axiom holds in H^p.

The continuity of addtion and scalar multiplication in H^p is a direct consequence of (3) and (4).

Theorem 3. 2 If $f \in H^p$, then

$$| f(z) | \leqq \| f \| / (1- | z |)^{1/p}, \quad | z | < 1$$

Proof. We may assume that $f \neq 0$

By virtue of *F.* Riesz's decomposition, we may write $f(z) = g(z) h(z)$, where $h(z)$ is regular and bounded by unity on $|z|<1$, and $g \in H^p$ with $\| g \| = \| f \|$ and $g(z) \neq 0$ on $|z| < 1$.

Thus it is clear that $[g(z)]^p$ may be defined so that it is a member of H^1.

By Cauchy's integral formula,

$$[g(z)]^p = \frac{1}{2\pi} \int_0^{2\pi} \frac{[g(\rho e^{i\theta})]^p \rho e^{i\theta} \, d\theta}{\rho e^{i\theta} - z}, \quad | z | < p < 1.$$

Thus
$$| g(z) |^p \leqq \frac{\rho}{\rho - | z |} \left(\frac{1}{2\pi} \int_0^{2\pi} | g(\rho e^{i\theta}) |^p \, d\theta \right)$$

$$\leqq \frac{\rho}{\rho - | z |} \cdot \| g \|^p = \frac{\rho}{\rho - | z |} \cdot \| f \|^p.$$

Thus, $| f(z) |^p \leqq (1/1- | z |) \cdot \| f \|^p$, the desired conclusion.

Theorem 3. 3 H^p is complete.

Proof. A proof of the completeness of H^p for $1 \leqq p$ is to be found in *A. E.* Taylor's paper and the proof is trivially modified for our case when $0 < p < 1$, when we note the Theorem 3. 2. M. Day has defined a topology over $L^p[0, 2\pi]$ which is eqivalent to the following: a set $U \subseteq L^p[0, 2\pi]$ is said to be open in case for arbitrary $f_0 \in U$ it is true that there exists some positive number $r > 0$

such that $E_r[\ \|f-f_0\|<r]$ lies in U, where for arbitrary $f\in L^p[0.2\pi]$, $\|f\| = ((1/2\pi)\cdot\int_0^{2\pi}|f(\theta)|^p d\theta)^{1/p}$.

That $L^p[0, 2\pi]$ is a linear topological space is seen just as it was that H^p is linear topological space.

Theorem 3.4 H^p is equivalent to a closed subspace of $L^p[0, 2\pi]$ (that is, there exists an algebraic isomorphism Γ *of* H^p onto a closed subspace of $L^p[0, 2\pi]$, and the isomorphism being norm preserving.)

Proof. F. Riesz has shown that if $f\in H^p$, then $\lim_{r\to 1} f(re^{it})$ exists almost everywhere on $[0, 2\pi]$ and moreover, if we designate $\lim_{r\to 1} f(re^{it})$ as $f(e^{it})$ we have $f(e^{it})\in L^p[0, 2\pi]$ and

$$\|f\| = \|f(e^{it})\|, \quad \text{that is,}$$

f has a member of H^p which has the same norm as $f(e^{it})$ has as a member of $L^p[0, 2\pi]$. We define $\Gamma(f)\equiv f(e^{it})$.

That Γ is one-to-one is clear from the norm preserving property of Γ and that the range of Γ is closed is clear since a cauchy sequence in the range of Γ maps into a cauchy sequence in H^p.

Theorem 3.5 H^p is perfectly seperable.

Proof. The proof that $L^p[0, 2\pi]$, $0<p<1$, is seperable is practically the same as the proof when $1\leq p<\infty$.

As in the case in any metric space, seperability in $L^p[0, 2\pi]$ implies perfect seperability. Whence, every subset of $L^p[0, 2\pi]$ is likewise perfectly seperable. Letting $R(\Gamma)$ be the range of Theorem 3.4, we see that $R(\Gamma)$ is perfectly seperable. Since Γ is a homeomorphism of H^p onto $R(\Gamma)$ then H^p is perfectly seperable.

4. Linear functional.

Although H^p is not normal in the case $p<1$, its bounded linear functionals can still be defined in the usual manner.

Thus a linear functionals f on H^p is said to be bounded if

$$\|f\| = \operatorname*{Sup}_{\|t\|_p=1} |f(t)| < \infty.$$

Since H^p is F-space even if $p<1$, the principle of uniform boundness theorem still applies ∴ every pointwise bounded linear functional on H^p is uniformly bounded. (See [1])

Let A denote the class of functions analytic in $|z|<1$ and continuous in $|z|\leqslant 1$.

It is convenient to write $f\in\Lambda_\alpha$ to indicate that $f\in A$ and its boundary function $f(e^{it})$ belongs to Lipschitz class Λ_α, $0<\alpha\leq 1$.

Theoren 4.1 To each bounded linear functional ϕ on H^p, $0<p<1$, there corresponds a unique function $g\in A$ such that

(1) $\phi(f)=\lim_{r\to 1}\frac{1}{2\pi}\int_0^{2\pi} f(re^{i\theta})\ g\ (e^{i\theta})d\theta,\ f\in H^p$

Proof. Given $\phi\in(H^p)^*$, let $b_k=\phi(z^k)$, $k=0,1,2\cdots\cdots$.

Then

$$|b_k|\leqslant\|\phi\|\cdot\|z^k\|=\|\phi\|.$$

So, the function

$$g(z)=\sum_{k=0}^{\infty}b_k z^k$$

is well-defined and analytic in $|z|<1$.

Suppose now that

$$f(z)=\sum_{k=0}^{\infty}a_k z^k\in H^p.$$

For fixed $0<p<1$, let $f_\rho(z)=f(\rho z)$.

Since f_ρ is the uniform limit on $|z|=1$ of the partial sums of its power series, and since ϕ is continuous, it follows that

$$\phi(f_\rho)=\lim_{N\to\infty}\phi\left(\sum_{k=0}^{N}a_k\rho^k z^k\right)=\sum_{k=1}^{\infty}a_k b_k\rho^k.$$

But $f_\rho\to f$ in H^p norm as $\rho\to 1$, so

(2) $\phi(f)=\lim_{\rho\to 1}\sum_{k=0}^{\infty}a_k b_k\rho^k.$

To deduse (1) from (2), it would suffices to show $g\in H^1$.

But for fixed ξ, $|\xi|<1$, let

$$f(z)=(1-\xi z)^{-1}=\sum_{k=0}^{\infty}\xi^k z^k.$$

Then by (2),

$$\phi(f)=\sum_{k=0}^{\infty}\xi^k b^k=g(\xi).$$

Hence

$$|g(z)|\leqslant\|\phi\|\ \|f\|_p\leqslant\|\phi\|\ \|(1-z)^{-1}\|_p,$$

that is, $g\in H^\infty$. And uniqueness is trivial.

Theorem 4.2 $\Gamma n, z\in(H^p)^*$, $n=0,1,2,3\cdots\cdots$, $|z|<1$, and

$$\|\Gamma o, z\|\leqslant\frac{1}{(1-z)^{1/p}}$$

$$\|\Gamma n, z\|\leqslant\frac{\rho_{n,z}}{(\rho_{n,z}-|z|)^{n+1}(1-\rho_{n,z})^{1/p}},\ n>0$$

where $\rho_{n,z} \equiv [\, |z|(1-p)+pn+\{[\,|z|(1-p)+pn]^2+4p\,|z|(pn+1)\}^{\frac{1}{2}}]/2(pn+1)$.

Proof. We may assume that $n>0$ since the case where $n=0$ is merely a direct consequence of Theorem 3.2.

Let $|z|<\rho<1$. Then, by Cauchy's integral formula for the derivatives of a regular function we have

$$\gamma_{n,z}(f)=(1/2\pi)\int_0^{2\pi} f(\rho e^{i\theta})\rho e^{i\theta}d\theta/(\rho e^{i\theta}-z)^{n+1}.$$

Whence, $|\gamma_{n,z}(f)|\le(\rho/(\rho-|z|))^{n+1}.\ (\|f\|/(1-\rho)^{1/p})$

by virtue of Theorem 3.2

Thus it is then clear that $\gamma_{n,z}\in(H^p)^*$ and if $|z|<\rho<1$, then

$$\|\gamma_{n,z}\|\le\rho/(\rho-|z|)^{n+1}(1-\rho)^{1/p}.$$

The final conclusion of our theorem is then obtained by minizing the function

$$\rho/(\rho-|z|)^{n+1}(1-\rho)^{1/p}$$

on the interval $|z|<\rho<1$.

Corollary 1. There are countable collection of linear functionals $\{\eta_n\}$ on H^p having the property that if $f\in H^p$ is not zero, then there exists n such that $\eta_n(f)=0$

Corollary 2. Suppose that F is a family of H^p such that $\gamma(f)$ is bounded on F for each fixed $\gamma(H^p)^*$. Then there exists $M\neq 0$ such that

(a) $|\gamma(f)|\leqq M\|\gamma\|$,

(b) $|f(z)|\leqq\dfrac{M}{(1-|z|)^{1/p}}$,

(c) $|\gamma_{n,z}(f)|\leqq M\{\dfrac{\rho_{n,z}}{(\rho_{n,z}-|z|)^{n+1}(1-\rho)^{1/p}}\}$, $n>0$,

for all $f\in F$

5. Main theorem

As in any normed linear space we say that the sequence $\{f_n\}\in H^p$ converges to $f\in H^p$, written $f_n\to^w f$, in case

$$\lim_{n\to\infty}\gamma(f_n)=\gamma(f)$$

for every $\gamma\in(H^p)^*$ where $(H^p)^*$ is a class of linear functionals on H^p.

Theorem 5.1 If $f_n\to^w f$ in H^p, then $\lim_{n\to\infty} f_n(z)=f(z)$ uniformly on all compact subsets of the unit circle.

Proof. Suppose $\gamma(f_n)\to\gamma(f)$ for all $\gamma\in(H^p)^*$, then $\{\gamma(f_n)\}$ is bounded in for each fixed $\gamma\in(H^p)^*$. By Corollary 2, since $|f_n(z)|\leqslant M/(1-|z|)^{1/p}$, $|f_n(z)|\leqslant M/(1-\rho)^{1/p}$ on $|z|\leqslant\rho<1$. Moreover, $\gamma_{0,z}(f_n)\to\gamma_{0,z}(f)$, or what is the same thing $f_n(z) \to f(z)$ on $|z| < 1$.

Thus, by Vitali's theorem $\lim_{n\to\infty} f_n(z) = f(z)$ uniformly on all compact subsets of $|z| < \rho$, where $\rho < 1$, and hence $\lim_{n\to\infty} f_n(z) = f(z)$ uniformly on all compact subsets of $|z| < 1$.

Theorem 5.2 Suppose $\{f_n\}$ is a bounded sequence of elements of H^p such that $\lim_{n\to\infty} \Gamma_{k,0}(f_n) = \Gamma_{k,0}(f)$ for all $k = 0, 1, 2, \cdots\cdots$.
Then $\lim_{n\to\infty} f_n(z) = f(z)$ uniformly on all compact subsets of the unit circle.

Proof. Let $\varepsilon > 0$ be given. Consider

$$|f_n(z) - f(z)| = \left|\sum_{k=0}^{\infty} \Gamma_{k,0}(f_n - f) z^k\right| \leqq \left|\sum_{k=0}^{N} \Gamma_{k,0}(f_n - f) z^k\right| + \left|\sum_{k=N+1}^{\infty} \Gamma_{k,0}(f_n - f) z^k\right|, \quad |z| \leqq \rho < 1.$$

By Theorem 4.2, we find that $\rho_{k,0} = pk/(pk+1)$, whence

$$\|\Gamma_{k,0}\| \leqq ((pk+1)/pk)^k (pk+1)^{1/p}.$$

It is easily verified that $\sum_{k=1}^{\infty} ((pk+1)/pk)^k. \ (pk+1)^{1/p} p^k$
converges.

We now choose N such that $M \sum_{k=N+1}^{\infty} ((pk+1)/pk)^k. \ (pk+1)^{1/p} p^k < \varepsilon/2$, where M is a bound on $\{\|f_n - f\|\}$.

Thus, since.

$$|\Gamma_{k,0}(f_n - f)| \leqq \|\Gamma_{k,0}\| \cdot \|f_n - f\| \leqq M((pk+1)/pk+1)/pk)^k (pk+1)^{1/p},$$

we find that $\left|\sum_{k=N+1}^{\infty} \Gamma_{k,0}(f_n - f) z^k\right| < \varepsilon/2$. We now choose $\bar{N}$ such that $n > \bar{N}$ implies $\left|\sum_{k=0}^{N} \Gamma_{k,0}(f_n - f) z^k\right| < \varepsilon/2$ for all z such that $|z| \leqq \rho$, as can well be accomplished since $\Gamma_{k,0}(f_n - f) \to 0$ by hypothesis. Thus, indeed $|f_n(z) - f(z)| \leqq \varepsilon/2 + \varepsilon/2 = \varepsilon$ on $|z| \leqq \rho$ for all $n > \bar{N}$ which concludes the proof of the theorem.

References

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[4] W. Rudin : *Real and Complex Analysis*, McGraw-Hill Book Company.