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A Remark on Noetherian Ring

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1. Introduction

Recently, the theory of commutative rings is blindingly developed and, in particular, the concepts of localization and completion of ring have been studied deeply.

In this paper, we shall investigate some relations among the Noetherian ring A, the localization of A and the $\mathfrak{a}$-adic completion of A. (See Theorem 3).

2. Localization and Completion of ring

Throughout this paper we assume that every ring is a commutative ring with identity.

Let S be a multiplicative subset of a ring A. Then the localization of A with respect to S is the ring

$$S^{-1}A=\{a/s \mid a\in A,\ s\in S\}$$

where equality is defined by

$$a/s=a'/s' \Longleftrightarrow s''(s'a-sa')=0 \text{ for some } s''\in S,$$

and the addition and the multiplication are defined by the usual formulas about fractions. The natural map $f: A\to S^{-1}A$ given by $f(a)=a/1$ is a homomorphism and its kernel is $\{a\in A \mid sa=0 \text{ for some } s\in S\}$. The A-algebra $S^{-1}A$ has the universal mapping property, and if A is Noetherian ring, so is $S^{-1}A$.

Let G be a topological abelian group. The topology of G is uniquely determined by the neighborhoods of 0 in G. A sequence (x_n) of elements of G is a Cauchy sequence if, for any neighborhood U of 0, there exists an integer $s(U)$ such that

$$x_n-x_m\in U \text{ for all } n, m\geqslant s(U).$$

Two Cauchy sequences are equivalent if $x_n-y_n\to 0$ in G. The set of all equivalence classes of Cauchy sequences is called the completion of G and is denoted by $\hat{G}$. If (x_n), (y_n) are Cauchy sequences, so is (x_n+y_n), and its class in $\hat{G}$

depends only on the classes of (x_n) and (y_n). Hence we have an addition in $\hat{G}$ with respect to which $\hat{G}$ is an abelian group. For each $x\in G$, the class of the constant sequence (x) is an element $\phi(x)$ of $\hat{G}$, and $\phi: G\longrightarrow\hat{G}$ is a homomorphism of abelian groups. These constructions show that $\phi: G\longrightarrow\hat{G}$ is injective iff G is Hausdorff. If $\phi: G\longrightarrow\hat{G}$ is an isomorphism we shall say that G is complete. Take $G=A$. subgroup $G_n=\mathfrak{a}^n$, where $\mathfrak{a}$ is an ideal in a ring A. The topology defined on A is called the $\mathfrak{a}$-adic topology, and the topology is Hausdorff iff $\cap\mathfrak{a}^n=(0)$. ([3], [5]). The completion $\hat{A}$ of A is again a topological ring ; $\phi: A\longrightarrow\hat{A}$ is a continuous ring homomorphism, whose kernel is $\cap\mathfrak{a}^n$. If A is a Noetherian ring, then the $\mathfrak{a}$-adic completion $\hat{A}$ of A is Noetherian. ([1], [2], [5], etc.).

Lemma 1. Let A be a Noetherian ring, $\mathfrak{a}$ an ideal, M a finitely-generated A-module and $\hat{M}$ the $\mathfrak{a}$-adic completion of M. Then the kernel $E=\bigcap_{n=1}^{\infty}\mathfrak{a}^n M$ of $M\longrightarrow\hat{M}$ consists of those $x\in M$ annihilated by some element of $1+\mathfrak{a}$.

Proof. Since E is the intersection of all neighborhoods of $0\in M$, the topology induced topology on it is trivial, *i. e.*, E is the only neighborhood of $0\in E$. And the induced topology on E coincides with its $\mathfrak{a}$-topology. ([3]). Since $\mathfrak{a}E$ is a neighborhood in the $\mathfrak{a}$-topology it follows that $\mathfrak{a}E=E$. Since M is finitely-generated and A is Noetherian, E is also finitely-generated and so there exists $a\equiv 1$ (mod $\mathfrak{a}$) such that $aE=0$. ([1]). Hence $1\text{-}a\in$ mod $\mathfrak{a}\Rightarrow 1\text{-}a=\alpha$ for some $\alpha\in\mathfrak{a}$.

$\Rightarrow 1\text{-}\alpha=a$ for some $\alpha\in\mathfrak{a}$.

i. e., $(1-\alpha)E=0$ for some $\alpha\in\mathfrak{a}$.

The converse is obvious : if $(1-\alpha)x=0$, then

$$x=\alpha x=\alpha^2 x=\cdots\in\bigcap_{n=1}^{\infty}\mathfrak{a}^n M=E.$$ Q. E. D.

3. Some Relations among the Noetherian ring A, the Localization of A and the Completion of A.

Lemma 2. Let A be a Noetherian ring, $\mathfrak{a}$ an ideal, $\hat{A}$ the $\mathfrak{a}$-adic completion of A and S the multiplicatively closed set $1+\mathfrak{a}$. Then the natural homomorphism $f: A\longrightarrow S^{-1}A$ and $g: A\longrightarrow\hat{A}$ have the same kernel.

Proof. Since $\ker g=\bigcap_{n=1}^{\infty}\mathfrak{a}^n$ and

$$\ker f=\{x\in A \mid x/1=0\}$$
$$=\{x\in A \mid (1+\alpha)x=0 \text{ for some } \alpha\in\mathfrak{a}\},$$

by Lemma 1, $\ker f=\ker g$. Q. E. D.

Theorem 3. Let A be a Noetherian ring, $\mathfrak{a}$ an ideal of A, $\hat{A}$ the $\mathfrak{a}$-adic completion of A and S the multiplicatively closed set $1+\mathfrak{a}$. Then

(i) $S^{-1}A$ is a subring of $\hat{A}$;

(ii) $S^{-1}A$ and $\hat{A}$ are flat A-algebra;

(iii) if $\mathfrak{a}$ is contained in the Jacobson radical of A, then $\hat{A}$ is a faithfully flat A-algebra.

Proof. (i) Since A is complete for its $\mathfrak{a}$-topology, for any $x\in\hat{\mathfrak{a}}$, $(1-x)^{-1}=1+x+x^2+\cdots$ converges in $\hat{A}$, so that $1-x$ is a unit, *i. e.*, every element of S becomes a unit in $\hat{A}$. ([2], [5], [6]). By the universal mapping property of $S^{-1}A$ this means that there exists a unique homomorphism

$$h: S^{-1}A \longrightarrow \hat{A} \text{ such that } g=h\circ f.$$

For any $a/s\in \ker h$,

$$\begin{aligned} h(a/s)=g(a)g(s)^{-1}=0 &\Rightarrow g(a)=0 \\ &\Rightarrow a\in\ker g=\ker f \text{ (by Lemma 2)} \\ &\Rightarrow f(a)=a/1=0 \\ &\Rightarrow ta=0 \text{ for some } t\in S \\ &\Rightarrow a/s=ta/ts=0. \end{aligned}$$

Hence h is injective. Therefore $S^{-1}A$ can be indentified with a subring of $\hat{A}$.

(ii) Since the operation S^{-1} is a exact and $S^{-1}A\cong A\otimes_A S^{-1}A$, $S^{-1}A$ is flat A-algebra. And if $0\to\mathfrak{a}\to A$ is exact, then $0\to\hat{\mathfrak{a}}\to\hat{A}$ is exact since the operation $\varprojlim$ is left exact. Since $\hat{\mathfrak{a}}\cong\mathfrak{a}\otimes_A\hat{A}$ and $\hat{A}\cong A\otimes_A\hat{A}$,

$$0\to\mathfrak{a}\otimes_A\hat{A}\to A\otimes_A\hat{A} \text{ is exact.}$$

Therefore $\hat{A}$ is flat A-algebra.

(iii) Let $\boldsymbol{m}$ be a maximal ideal of A. Then $\boldsymbol{m}\supseteqq\mathfrak{a}$, hence $\boldsymbol{m}$ is open in A and so

$$\hat{A}/\boldsymbol{m}\hat{A}\cong A/\boldsymbol{m}.$$

Thus $\boldsymbol{m}\hat{A}\neq\hat{A}$. Since $\hat{A}$ is flat over A by (ii), $\hat{A}$ is a faithfully flat over A.

Q. E. D

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