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A Note on Generalizations of *PCI*-rings

By Si-Joo Kim

1. Introductioı

In [1], rings whose cyclic left R-modules not isomorphic to $_RR$ are injective (called left *PCI*-rings) are considered. A left R-module M is called *CF*-injective if for any finitely generated left R-module F, any cyclic left submodule C of F, every left R-homomorphism of C into M extends to one of F into M. A left R-module M is called *MP*-injective if for any principal left ideal P of R, any left R-monomorphism of P into M extends to a left R-homomorphism of R into M. In [7], *CF*-injectivity and *MP*-injectivity, generalizations of injectivity are introduced and connections between injectivity, *CF*-injectivity, *MP*-injectivity and Von Neumann regularity are found.

In this note, we introduce *PCM*-ring and *PCF*-ring, generalizations of *PCI*-ring and prove that if R is left self-injective and every singular left R-module is injective, then a left *PCM*-ring R is left V-ring. If R is a directly finite left *PCF*-ring, then either R is a regular ring or a left semihereditary simple domain. Furthermore, a ring R is finitely embedded left *PCF*-ring iff R is a semisimple.

Throughout this note, R means an associative ring with identity and every left R-module is unitary. If a left R-module M is essential extension of a left R-module N, we write $N \leqq_e M$ to denote this situation.

2. *PCM*-rings and *PCF*-rings

We introduce the following generalizations of *PCI*-rings.

Definition (1) If every cyclic left R-module C not isomorphic to $_RR$ is *MP*-injective, then R is called a left *PCM*-ring.

(2) A ring is a left *PCF*-ring if every cyclic left R-module not isomorphic to $_RR$ is *CF*-injective.

Let R be a left non-singular ring such that $_RR$ is not finite dimensional. Then injective hull $E(R)$ of R is a regular ring but not semisimple ring. By [proposition 6.

11, 1], $E(R)$ is not a left PCI-ring but it is a left PCM-ring by proposition 2. 2 below.

Lemma 2. 1. If a cyclic submodule C of a finitely generated left R-module M is CF-injective, then C is a direct summand of M. Moreover, a principal MP-injective left ideal of R is direct summand of R.

Proof. Consider identity map $1_C : C \to C$. Since C is CF-injective, 1_C extends to a homomorphism $f : M \to C$ and $fi = 1_C$ where inclusion map $i : C \subset M$. Thus C is a direct summand of M. The latter part can be shown by similar method.

Corollary. If a cyclic submodule C of a finitely generated left R-module M is a direct summand of some CF-injective submodule of M, then C is a direct summand of M.

Proposition 2. 2. Every cyclic left R-module is MP-injective iff R is a (Von Neumann) regular ring.

Proof. It is sufficient to show that every left R-module is MP-injective by [7]. Let M be a left R-module. Consider a monomorphism $f : P \to M$ for any principal left ideal P of R. Since P is MP-injective, the identity map $1_P : P \to P$ extends to a homomorphism $g : R \to P$. Then $fg : R \to M$ is an extension of f.

Corollary. Every essential left ideal of R is MP-injective iff every principal left ideal of R is MP-injective iff R is a regular ring.

Proof. For any principal left ideal Rx of R for $x \in R$, there exists a left ideal L of R such that $Rx \oplus L \leqq_e {}_RR$. Hence Rx is MP-injective and a direct summand of R by lemma 2. 1.

Proposition 2. 3. A left PCM-ring R is left non-singular.

Proof. Consider a principal left ideal Rx of R. If Rx is not isomorphic to ${}_RR$, then Rx is MP-injective. By lemma 2. 1, Rx is a direct summand of R. Consider an exact sequence

$$0 \to 1(x) \to R \to Rx \to 0 \quad \text{where } l(x) = \{ r \in R \mid rx = 0 \}.$$

Since Rx is projective for any $x \in R$, $R = l(x) \oplus J$ for some left ideal J of R. If $x \neq 0$, $l(x) \neq R$ and $J \neq 0$. Since $l(x) \cap J = 0$, $l(x) \nleqq_e R$. Thus x is not contained in the left singular ideal of R.

Proposition 2. 4. Let R be a left self-injective ring and let every singular left R-module be injective. Then left PCM-ring R is a left V-ring.

Proof. Let M be a simple left R-module. If M is isomorphic to R, M is injective left R-module. If M is not isomorphic to R, then M is MP-injective. Consider a non-zero homomorphism $f: L \to M$ for any essential left ideal L of R. Let B be a left ideal of R in L maximal with respect to the property $B \cap Ker(f) = 0$. If $B \neq 0$, the restriction map $f|_{Rx}: Rx \to M$ is an isomorphism for some non-zero $x \in B$. Hence Rx is MP-injective. By lemma 2. 1, Rx is a direct summand of R and hence injective. If $B = 0$, $Ker(f) \leqq_e L$. Since ${}_RR$ is non-singular by proposition 2. 3, L is non-singular and $L/Ker(f)$ is singular. Since M is isomorphic to $L/Ker(f)$, M is injective. Thus every simple left R-module is injective and R is a left V-ring.

Proposition 2. 5. If R is a left PCF-ring containing a non-trivial central idempotent, then R is a regular ring.

Proof. If e is a non-trivial central idempotent in R, then neither Re nor $R(1-e)$ is isomorphic to ${}_RR$. Thus $R = Re \oplus R(1-e)$ is CF-injective since a finite direct sum of CF-injective left R-modules is CF-injective. Thus every principal left ideal of R is CF-injective and hence a direct summand of R by lemma 2. 1. Therefore R is a regular ring.

Proposition 2. 6. (1) If R is a left PCF-ring, then R is semi-prime.

(2) If R is a left PCF-domain, then R is left semihereditary.

Proof. (1) For any $a(\neq 0) \in R$, if Ra is not isomorphic to ${}_RR$, Ra is CF-injective and hence a direct summand of R. Thus there exists a non-zero idempotent element $e \in R$ such that $Ra = Re$. Hence we have $(Ra)^2 \neq 0$. If there exists an isomorphism $f: Ra \to {}_RR$, we can find $c \in Ra$ such that $f(c) = 1$. Then $l(c) = 0$ and this implies $0 \neq Rac \subset (Ra)^2$. This proves that R is semi-prime.

(2) Let P be any non-zero projective left ideal of R and $a \in R$. If $C = Ra + P$ is such that C/P is isomorphic to ${}_RR$, then there exists $x(\neq 0) \in R$ such that $l(x) = \{ r \in R \mid ra \in P \}$. Since R is domain, this implies that $Ra \cap P = 0$ and hence $C = Ra \oplus P$. If Ra is not isomorphic to ${}_RR$, then Ra is a direct summand of projective ${}_RR$ and hence C is projective.

If C/P is not isomorphic to ${}_RR$, then C/P is CF-injective. Lemma 2. 1. shows that C/P is a direct summand of R/P. Let D be a left ideal of R containing P such that D/P is a relatively complement of C/P in R/P. Then $R = C + D$ and $P = C \cap D$. Hence we have an exact sequence

$0 \to P \to C \oplus D \to R \to 0$. Thus $C \oplus D$ is isomorphic to projective left R-module $P \oplus R$. Hence C is projective. Since any principal left ideal of R is projective, induction on the number of generators of P shows that any finitely generated left ideal of R is projective.

Corollary. If R is a left PCF-ring with finitely generated non-zero left (right) socle, then R is a regular ring.

Proof. Since R is semi-prime by proposition 2.6, its left socle is equal to its right socle. Since the socle $Soc(R)$ of R is two-sided ideal of R, $Soc(R)$ is generated by a central idempotent element of R. If $R=Soc(R)$, clearly R is regular. If $R\neq Soc(R)$, $Soc(R)$ is generated by a non-trivial central idempotent element. By proposition 2.5 R is a regular ring.

Proposition 2.7. If R is a left PCF-ring, then R is a left V-ring.

Proof. It is sufficient to show that every simple left R-module is CF-injective by [7]. Let M be a simple left R-module. If M is not isomorphic to $_RR$, then M is CF-injective. If M is isomorphic to $_RR$, then simple R as a left R-module is a left self-injective. Hence M is CF-injective.

Recall that R is directly finite if $xy=1$ implies $yx=1$ for any $x, y\in R$. Then R is directly finite iff $_RR\oplus {_RM}$ is isomorphic to $_RR$ implies $M=0$.

Proposition 2.8. Let R be a directly finite left PCF-ring. Then R is either a regular ring or a left semihereditary simple domain.

Proof. If R is a domain, R is left semihereditary and V-domain by proposition 2.7. Also R is simple. Suppose that R is not a domain. Then there exists a non-zero element $x\in R$ such that $l(x)\neq 0$. It is easy to show that Rx is projective. Thus $l(x)=Re$ for some non-trivial idempotent $e\in R$. Since R is directly finite, Re and $R(1-e)$ must be CF-injective. Hence $R=Re\oplus R(1-e)$ is CF-injective. Therefore every principal left ideal of R is a direct summand of R which implies that R is regular.

3. Finitely embedded modules

A left R-module M is essentially finitely generated if M has a finitely generated essential submodule. In particular, when $Soc(M)$ of a left R-module M is a finitely generated essential submodule of M, M is called a finitely embedded module.

Proposition 3.1. If B is a closed submodule of a finitely embedded left R-module M, then M/B is finitely embedded.

Proof. Let B be a closed submodule of finitely embedded left R-module. Since $Soc(M) \leqq_e M$, $B+Soc(M) \leqq_e M$ and $(B+Soc(M))/B \leqq_e M/B$. Let $p: M \to M/B$ be the projection. Since $p(Soc(M)) \subset Soc(M/B)$, $Soc(M/B) = (B+Soc(M))/B$ and $(B+Soc(M))/B$ is finitely generated. Hence M/B is finitely embedded.

Corollary. Let C be a cyclic CF-injective submodule of a finitely generated, finitely embedded left R-module M. Then M/C is finitely embedded.

Proposition 3. 2. If a left R-module M is essentially finitely generated and every finitely generated submodule of M is finitley embedded, then M is finitely embedded.

Proof. Since M is essentially finitely generated, there exists a finitely generated submodule N of M such that $N \leqq_e M$. Thus N is finitely embedded, i. e., $Soc(N)$ is finitely generated and $Soc(M) = Soc(N) \leqq_e N \leqq_e M$. Hence M is finitely embedded.

Lemma 3. 3. A proper essential submodule of a left R-module M is finitely embedded iff M is finitely embedded.

Proof. M is finitely embedded iff $Soc(M)$ is finitely generated and $Soc(M) \leqq_e M$. Let S be a proper finitely embedded essential submodule of M. Then $Soc(S) \leqq_e S \leqq_e M$ and $Soc(S) = Soc(M)$ is finitely generated.

Proposition 3. 4. A ring R is finitely embedded left PCF-ring iff R is semisimple.

Proof. Let L be an essential left ideal of R. Since R is finitely embedded, $Soc(L) = Soc(R) \leqq_e L \leqq_e R$ and $Soc(L)$ is finitely generated. Since R is a regular ring by corollary of proposition 2. 6, $Soc(L)$ is a direct summand of ${}_R R$. Hence $Soc(L) = R$. Therefore, R is semisimple. Since R is semisimple iff every cyclic left R-module is injective, the converse part is trivial.

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An-Dong National College