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Some Characterizations on Generic Submanifolds of a Complex Projective Space

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§ 0. Introduction

Let S^{2m+1} be the unit hypersphere in a $(m+1)$-dimensional complex number space C^{m+1} naturally identified with $R^{2(m+1)}$. It is, then, a principal circle bundle over a complex projective space CP^m and the Riemannian structure on CP^m is given by the natural projection $\tilde{\pi}: S^{2m+1}\to CP^m$ that is defined by the Hopf-fibration $S^1\to S^{2m+1}\to CP^m$, which is the Riemannian submersion with totally geodesic fibres.

Since the model spaces $M^c_{p,q}(a,b)=\tilde{\pi}(S^{2p+1}(a)\times S^{2q+1}(b))$ immersed in CP^m, where (p,q) is some portion of $m-1$ and $a^2+b^2=1$, have been well known to us, many authors have studied sufficient conditions and necessary and sufficient conditions to be one of model spaces $M^c_{p,q}(a,b)$ (cf. Lawson [3], Maeda [4], Okumura [5]).

Recently these notions are mainly considered in the generic submanifolds or CR-submanifolds of CP^m by many authors (cf. Ki [2], Kim [2], Kon [9], Pak [6], Yano [9]). In particular, Pak and present author (cf. [7]) studied another necessary and sufficient conditions, which are concerned with the locally symmetry of $\tilde{\pi}^{-1}(M^c_{p,q}(a,b))$, to be one of model space $M^c_{p,q}(a,b)$ by using the theory of the Riemannian fibre bundles. The generalization of these facts to the generic submanifolds of CP^m is the purpose of this present paper. Thus, in this paper we will discuss some necessary and sufficient conditions also concerned with the locally symmetry and Ricci-parallel of $S^{p_1}(r_1)\times\cdots\times S^{p_N}(r_N)$. And we will study some pinching problems in the class of $\tilde{\pi}(S^{p_1}(r_1)\times\cdots\times S^{p_N}(r_N))$ by using the following Theorem A.

Theorem A. (Ki, Pak and Kim [2]) *Let M be an n-dimensional complete generic submanifold of a complex projective space CP^m with flat normal connection. If the f-structure induced on M is normal and if the mean curvature vector is* *parallel in the normal bundle, then M is of the form*

$$\tilde{\pi}(S^{P_1}(r_1)\times\cdots\times S^{P_N}(r_N)),\quad P_1,\cdots,P_N \text{ are odd numbers}$$

$$P_1+P_2+\cdots+P_N=n+1,\quad r_1^2+r_2^2+\cdots+r_N^2=1,\quad N=2m-n+1,$$

where $S^{P_i}(r_i)$ is a P_i-dimensional sphere with radius r_i.

Manifolds, gometric objects and mappings we discuss in this paper are assume to be differentiable and of C^∞. We will use the following system of indices in this paper

$$\kappa,\mu,\nu,\lambda=1,2,\cdots\cdots,2m+1 \ :\ h,i,j,k=1,2,\cdots\cdots,2m,$$

$$\alpha,\beta,\gamma,\delta=1,2,\cdots\cdots,n+1 \ :\ a,b,c,d=1,2,\cdots\cdots,n,$$

$$u,v,w,x,y,z=1,2,\cdots\cdots\cdots,p \quad (n+p=2m)$$

The summation convention will be used with respect to those system indices.

§ 1. Generic submanifolds of a complex projective space.

As is well known, S^{2m+1} admits Sasakian structure $\{\tilde{\phi},\tilde{\xi},\tilde{g}\}$ and each fibre $\tilde{\pi}^{-1}(x)$ of $x\in CP^m$ is a maximal integral submanifold of the distribution spanned by $\tilde{\xi}$. The base space CP^m thereby admits the induced Kaehlerian structure of constant holomorphic sectional curvature $c=4$. Thus, if we let CP^m be covered by a system of coordinate neighborhoods $\{\tilde{U}:y^j\}$ and denote by g_{ji} components of the Hermitian metric tensor and by F_i^j those of almost complex structure of CP^m. Then we have

(1.1) $$F_h^iF_j^h=-\delta_j^i,\qquad F_j^hF_i^kg_{hk}=g_{ji}$$

and denoting by $\tilde{\nabla}_j$ the operator of the covariant differentiation with respect to g_{ji}, then we get

(1.2) $$\tilde{\nabla}_jF_i^h=0.$$

Let's denote by $K_{kji}{}^h$ components of the curvature tensors of CP^m.
The consistancy of the holomorphic sectional curvature of CP^m, then, gives

(1.3) $$K_{kji}{}^h=\delta_k^hg_{ji}-\delta_j^hg_{ki}+F_k{}^hF_{ji}-F_j{}^hF_{ki}-2F_{kj}F_i{}^h.$$

Let M be an n-dimensional Riemannian manifold covered by a system of coordinate neighborhoods $\{U:x^a\}$ and suppose now that it is immersed isometrically in CP^m by the following parametric equations

(1.4) $$y^j=y^j(x^a).$$

We put $B_c^j=\partial_cy^j$ $(\partial_c=\partial/\partial x_c)$ and denote by C_x^h p-mutually orthogonal unit normal vector fields on M. Then the first fundamental tensor g_{cb} which is Riemannian metric of M is given by $g_{cb}=B_c^jB_b^ig_{ji}$ because the immersion is isometric.

Denoting ∇_b the operator of van der Waerden-Bortolotti covariant differentiation with respect to g_{cb}, then we have equation of Gauss and Weingarten

(1.5) $\nabla_c B_b^j = A_{cb}^{\ x} C_x^j, \quad \nabla_c C_x^j = -A_{c\ x}^{\ b} B_b^j$

respectively, where $A_{cb}^{\ x}$ is the second fundamental tensors with respect to normals C_x^j and $A_{c\ x}^{\ a} = A_{cbx}\ g^{ab} = A_{cay} g^{ab} g_{xy}$, g_{xy} being the metric tensor of the normal bundle of M given by $g_{xy} = C_x^j C_y^i g_{ji}$ and $(g^{ba}) = (g_{ba})^{-1}$.

For a generic submanifold M of CP^n, we have equations of the form

(1.6) $F_j^{\ h} B_c^j = f_c^{\ a} B_a^h + f_c^{\ x} C_x^h, \qquad F_j^{\ h} C_x^j = -f_x^{\ a} B_a^h,$

where $f_c^{\ a}$ is a tensor field of type (1, 1) defined on M, $f_c^{\ x}$ that of mixed type and $f_x^{\ a} = f_c^{\ y} g^{ca} g_{yx}$.

If we apply $F_h^{\ j}$ to (1.6) and use (1.1) and (1.6) itself, we may obtain

(1.7) $f_a^{\ b} f_b^{\ c} = -\delta_a^c + f_a^{\ x} f_x^{\ c}, \quad f_a^{\ b} f_b^{\ y} = 0, \quad f_x^{\ a} f_a^{\ b} = 0, \quad f_x^{\ a} f_a^{\ y} = \delta_x^y,$

which means that the aggregate $(f_a^{\ b},\ f_a^{\ x},\ f_x^{\ b})$ is what we call f-structure satisfying $f^3 + f = 0$.

Using the fact that $F_{ji} = -F_{ij}$, $F_j^{\ h} g_{hi} = F_{ji}$, from (1.6) we have that

(1.8) $f_{cb} = -f_{bc}, \qquad f_{cx} = f_{xc}$

where we have put $f_{cb} = f_c^{\ a} g_{ab}$, $f_{bx} = f_b^{\ y} g_{yx}$ and $f_{xb} = f_x^{\ a} g_{ab}$.

Applying the operator $\nabla_a = B_a^i \tilde{\nabla}_i$ to (1.6) and using (1.2), (1.5), we can easily verify that

(1.9) $\nabla_b f_a^{\ c} = A_{b\ x}^{\ c} f_a^{\ x} - A_{ba}^{\ \ x} f_x^{\ c},$

(1.10) $\nabla_b f_a^{\ x} = -A_{bc}^{\ \ x} f_a^{\ c}, \quad \nabla_b f_x^{\ a} = A_{b\ x}^{\ c} f_c^{\ a}, \quad A_{ba}^{\ \ y} f_x^{\ a} = A_{b\ x}^{\ a} f_a^{\ y}.$

Since ambient manifold CP^n has constant holomorphic sectional curvature 4, using (1.3), then the structure equations of Gauss Codazzi and Ricci are respectively given by

(1.11) $K_{dcb}^{\ \ \ \ a} = \delta_d^a g_{cb} - \delta_c^a g_{db} + f_d^{\ a} f_{cb} - f_c^{\ a} f_{db} - 2 f_{dc} f_b^{\ a} + A_{d\ x}^{\ a} A_{cb}^{\ \ x} - A_{c\ x}^{\ a} A_{db}^{\ \ x},$

(1.12) $\nabla_d A_{cb}^{\ \ x} - \nabla_c A_{db}^{\ \ x} = f_d^{\ x} f_{cb} - f_c^{\ x} f_{db} - 2 f_{dc} f_b^{\ x},$

(1.13) $K_{dcy}^{\ \ \ \ x} = f_d^{\ x} f_{cy} - f_c^{\ x} f_{dy} + A_{de}^{\ \ x} A_{c\ y}^{\ e} - A_{ce}^{\ \ x} A_{d\ y}^{\ e},$

where we denote $K_{dcb}^{\ \ \ \ a}$ and $K_{dcy}^{\ \ \ \ x}$ by the components of the curvature tensors determined by the induced metric f_{cb} and g_{yx} in M and in the normal bundle of M respectively.

We now introduce a tensor field S of type (1, 2) of the following form

$$S_{cb}^{\ \ a} = [f, f]_{cb}^{\ \ a} + (\nabla_c f_b^{\ x} - \nabla_b f_c^{\ x}) f_x^{\ a},$$

where $[f, f]_{cb}^{\ \ a} = f_c^{\ e} \nabla_e f_b^{\ a} - f_b^{\ e} \nabla_e f_c^{\ a} - (\nabla_c f_b^{\ e} - \nabla_b f_c^{\ e}) f_e^{\ a}$ is the Nijenhuis tensor formed with f_c^a. Moreover, when S_{cb}^a vanishes identically, the induced f-structure is said to be normal. Thus, we may have the following.

Proposition 1.1 (See [2]) *Let M be a generic submanifold of a Kaehlerian manifold M. In order for the f-structure induced on M to be normal, it is necessary and sufficient the that second fundamental tensors A_{cb}^x and f_c^a commute.*

§ 2. Some properties concerned with the locally symmetry.

We consider a fibration $\pi: \bar{M} \to M$ which is compatible with the Hopf-fibration $\tilde{\pi}: S^{2m+1} \to CP^m$, where M is an n-dimensional generic submanifold of CP^m and $\bar{M} = \tilde{\pi}^{-1}(M)$ is a generic submanifold of S^{2m+1}. More precisely speaking, $\pi: \bar{M} \to M$ is a fibration with totally geodesic fibres such that the following diagram commutative:

$$(2.1) \qquad \begin{array}{ccc} \bar{M} & \xrightarrow{\tilde{i}} & S^{2m+1} \\ \downarrow & & \downarrow \\ M & \xrightarrow{i} & CP^m \end{array}$$

where $\tilde{i}: M \to S^{2m+1}$ and i : $M \to CP^m$ are isometric immersions.

Now we will give an example that satisfies the above commutative diagram (2.1). Consider the complex number space $C^{(p_i+1)/2}$ which can be naturally embedded as a totally geodesic sumanifold of C^{m+1}. Then C^{m+1} is identified with the product space $C^{(p_1+1)/2} \times C^{(p_2+1)/2} \times \cdots\cdots \times C^{(p_N+1)/2}$, where $p_1 + \cdots\cdots + p_N = n+1$, $N = 2m-n+1$ If we denote $S^{p_i}(r_i)$ by a P_i-dimensional sphere with radius r_i, then it becomes a hyperspere of $C^{(p_i+1)/2}$. Since S^{2m+1} is the unit hypersphere of C^{m+1}, the product space $S^{p_1}(r_1) \times S^{p_2}(r_2) \times \cdots \times S^{p_N}(r_N)$ of hyperspheres $S^{p_i}(r_i)$ $(i = 1, \cdots, N)$ may be considered as a submanifold of S^{2m+1} for $r_1^2 + r_2^2 + \quad + r_N^2 = 1$. On the other hand, Yano and Kon [9] showed that $S^{p_1}(r_1) \times \cdots\cdots \times S^{p_N}(r_N)$ is a generic submanifold of S^{2m+1} with parallel mean curvature vector and flat normal connection. Clearly we now have that $\pi(S^{p_1}(r_1) \times \cdots\cdots \times S^{p_N}(r_N))$ projected by the compatible fibration π is also an example of a generic submanifold which satisfies above commutative diagram (2.1).

Let S^{2m+1} be covered by a system of coordinate neighborhoods $\{\hat{U} : y^x\}$ such that $\tilde{\pi}(\hat{U}) = \tilde{U}$ are coordinate neighborhoods of CP^m with local coordinate (y^j). Then we may represent the projection $\tilde{\pi}$ by $y^j = y^j(y^x)$ and put $E_x^j = \partial_x y^j$ $(\partial_x = \partial/\partial y^x)$ with the rank of the matrix (E_x^j) being always $2m$. Let's denot by $\tilde{\xi}^x$ components of ξ the unit Sasakian structure of S^{2m+1} induced from C^{m+1}. Then $\{E_x^j,$

$\tilde{\xi}_x\}$ becomes a local coframe in $\tilde{U}$, where $\tilde{\xi}_x = \tilde{\xi}^\mu g_{\mu x}$, being components of the metric tensor on S^{2m+1}. If we define $E^x{}_j$ by $(E^x{}_j, \tilde{\xi}^x) = (E_x{}^j, \tilde{\xi}_x)^{-1}$, then $\{E^x{}_j, \tilde{\xi}\}$ is a local frame in $\hat{U}$, which is dual to $\{E_x{}^j, \tilde{\xi}_x\}$.

We now take coordinate neighborhoods $\{\bar{U} : x^a\}$ of $\bar{M}$ such that $\pi(\bar{U}) = U$ are coordinate neighborhoods of M with local coordinates (x^a). If we denote the immersion $\bar{i}$ and submersion π by $y^x = y^x(x^a)$ and $x^a = x^a(x^a)$ respectively, then the commutative diagram (3. 1) implies that

$$B_a{}^j E_a{}^a = E_x{}^j B_a{}^x, \qquad E^x{}_j B_b{}^j = B_a^x E^a{}_b$$

where $E_a{}^a = \partial_a x^a$ and $B_a{}^x = \partial_a y^x$. Hence the Sasakian structure vector $\tilde{\xi}$ is always tangent to $\bar{M}$.

Putting by ξ^a components of $\tilde{\xi}$ in a coordinate neighborhood $\{\bar{U} : x^a\}$ of $\bar{M}$, we may similarly obtain a local frame $\{E_a{}^a, \xi^a\}$ and its dual coframe $\{E_a{}^a, \xi_a\}$ in $\bar{U}$, where ξ_a is the associated vector field of ξ^a with respect to the metric tensors $g_{\beta a} = \tilde{g}_{\mu x} B_\beta{}^\mu B_a{}^x$ of $\bar{M}$.

Since the metrics $\tilde{g}_{\lambda\mu}$ on S^{2m+1} and $g_{\beta a}$ on $\bar{M}$ are invariant with respect to the submersion $\tilde{\pi}$ and π respectively, the van der Waerden-Borotolotti covariant derivatives of $E_\lambda{}^i$, $E^\lambda{}_i$ and $E_a{}^a$, $E^a{}_a$ are respectively given by (cf. Ishihara and Konishi [1])

$$(2.2) \qquad \tilde{D}_\mu E_\lambda{}^i = -F_j{}^i (E_\mu{}^j \tilde{\xi}_\lambda + \tilde{\xi}_\mu E_\lambda{}^j), \quad \tilde{D}_\mu E^\lambda{}_i = -F_{ji} E_\mu{}^j \tilde{\xi}^\lambda + F_i{}^h \tilde{\xi}_\mu E^\lambda{}_h,$$

$$(2.3) \qquad \nabla_\beta E_a{}^a = -f_b{}^a (E_\beta{}^b \xi_a + \xi_\beta E_a{}^b), \qquad \nabla_\beta E^a{}_a = -f_{ba} E_\beta{}^b \xi^a + f_a{}^b \xi_\beta E^a{}_b,$$

where $\tilde{D}_\mu$ and ∇_β denote the operators of covariant differentiations with respect to $g_{\lambda\mu}$ and $g_{\beta a}$ respectively. Moreover, equations of co-Gauss, of co-Codazzi and of co-Ricci for the compatible submersion π are respectively given by the following forms

$$(2.4) \qquad \bar{K}_{dcb}{}^a = K_{dcb}{}^a - f_d{}^a f_{cb} + f_c{}^a f_{ab} + 2 f_{dc} f_b{}^a,$$

$$(2.5) \qquad \bar{K}_{dcb}{}^0 = -(\nabla_d f_{cb} - \nabla_c f_{db}),$$

$$(2.6) \qquad \bar{K}_{0cb}{}^0 = f_{ce} f_b{}^e,$$

where we have put

$$(2.7) \qquad \bar{K}_{dcb}{}^a = \bar{K}_{\delta\gamma\beta}{}^a E^\delta{}_d E^\gamma{}_c E^\beta{}_b E_a{}^a,$$

$$(2.8) \qquad \tilde{K}_{dcb}{}^0 = \tilde{K}_{\delta\gamma\beta}{}^a E^\delta{}_d E^\gamma{}_c E^\beta{}_b \xi_a \text{ and}$$

$$(2.9) \qquad \bar{K}_{0cb}{}^0 = \bar{K}_{\delta\gamma\beta}{}^a \xi^\delta E^\gamma{}_c E^\beta{}_b \xi_a,$$

$\bar{K}_{\delta\gamma\beta}{}^a$ being components of the curvature tensor of $\bar{M}$ and $\bar{K}_{dcb}{}^a$ being projectable local functions defined on $\{\bar{U} : x^a\}$.

When $\bar{M}$ is a locally symmetric space as a generic submanifold of S^{2m+1}, taking covariant derivative ∇_c to $\tilde{K}_{dcb}{}^a = K_{\sigma\gamma\beta}{}^\alpha E^\sigma{}_d E^\gamma{}_c E^\beta{}_b E_\alpha{}^a$ gives

$$\nabla_\varepsilon \tilde{K}_{dcb}{}^a = K_{\sigma\gamma\beta}{}^\alpha \{(\nabla_\varepsilon E^\sigma{}_d) E^\gamma{}_c E^\beta{}_b E_\alpha{}^a + E^\sigma{}_d (\nabla_\varepsilon E^\gamma{}_c) E^\beta{}_b E_\alpha{}^a + E^\sigma{}_d E^\gamma{}_c (\nabla_\varepsilon E^\beta{}_b) E_\alpha^a + E^\sigma{}_d E^\gamma{}_c E^\beta{}_b (\nabla_\varepsilon E_\alpha{}^a)\},$$

from which and using (2.3), (2.7) and (2.8) imply that

$$(2.10)\quad \nabla_\varepsilon K_{dcb}{}^a = (f_d{}^e K_{ecb}{}^a + f_c{}^e K_{deb}{}^a + f_b{}^e K_{dce}{}^a - K_{dcb}{}^e f_e{}^a)\xi_\varepsilon - (f_{ed} K_{ecb}{}^a + f_{ec} K_{deb}{}^a + f_{eb} K_{dce}{}^a + f_e{}^a K_{dcb e}) E_\varepsilon^e.$$

Transvecting ξ^ε to (2.10) first and taking account of projectability of local function $\tilde{K}_{dcb}{}^a$, that is $(\mathcal{L}_\xi K^H)^H = 0$, we may obtain

$$(2.11)\quad f_d{}^e \tilde{K}_{ecb}{}^a + f_c{}^e \tilde{K}_{deb}{}^a + f_b{}^e \tilde{K}_{dce}{}^a - f_e{}^a \tilde{K}_{dcb}{}^e = 0,$$

because we have used $\tilde{\xi}^\kappa = \xi^\alpha B_\alpha{}^\kappa$, and $\xi^\varepsilon E_\varepsilon = 0$, $\xi^\varepsilon \xi_\varepsilon = 1$. Using (2.4) to (2.11), then we easily get

$$(\mathrm{I})\qquad f_d{}^e K_{ecb}{}^a + f_c{}^e K_{deb}{}^a + f_b{}^e K_{dce}{}^a - f_e{}^a K_{dcb}{}^e = 0.$$

Next, transvecting $E^\varepsilon{}_f$ to (2.10) and making use of (1.9) and (2.5) to thus obtained equation, consequently we find

$$(\mathrm{II})\qquad (f_{ed} f_b{}^x + f_{eb} f_d{}^x + \nabla_e A_{db}{}^x) A_{cax} + (f_{ec} f_a{}^x + f_{ea} f_c{}^x + \nabla_e A_{ca}{}^x) A_{dbx} - (f_{ed} f_a{}^x + f_{ea} f_d{}^x + \nabla_e A_{da}{}^x) A_{cbx} - (f_{ec} f_b{}^x + f_{eb} f_c{}^x + \nabla_e A_{cb}{}^x) A_{dax} = 0.$$

where ∇_e denote the operator of covariant differentiation with respect to g_{cb}. Similarly also taking covariant derivative to (2.8) and making use of (2.3), (2.5) and (2.6), we easily get

$$(2.12)\quad \nabla_\varepsilon \tilde{K}_{dcb}{}^0 = (f_d{}^e \tilde{K}_{ecb}{}^0 + f_c{}^e \tilde{K}_{deb}{}^0 + f_b{}^e \tilde{K}_{dce}{}^0)\xi_\varepsilon - (f_{ed} \tilde{K}_{0cb}{}^0 + f_{ec} \tilde{K}_{d0b}{}^0 - f_{ea} \tilde{K}_{dcb}{}^a) E_\varepsilon{}^e,$$

from which, transvecting $E^\varepsilon{}_f$ and using (2.4), (2.6), we have

$$(\mathrm{III})\qquad f_{de} f_{bx} f_c{}^x - f_{ce} f_{dx} f_b{}^x - f_{ex}(A_d{}^a{}_x A_{cb}{}^x - A_c{}^a{}_x A_{db}{}^x) = \nabla_e \nabla_d f_{cb} - \nabla_e \nabla_c f_{db}.$$

If we finally transvect (2.12) with ξ^ε, then we may obtain

$$(\mathrm{IV})\qquad f_c{}^e A_{eb}{}^x f_x{}^b = 0,$$

where we have used (1.9), (2.5) and the projectability of $\tilde{K}_{dcb}{}^0$ because of $\mathcal{L}_\xi \nabla_d f_{cb} = 0$.

For a real hypersurface M of CP^m, Maeda [4] showed that a complete real hypersurface M satisfying (I) and (IV) is congruent to $M^c_{p,q}(a,b) = \pi(S^{2p+1}(a) \times S^{2q+1}(b))$. Moreover, Pak and present author [7] proved that M satisfying (II) and (IV), or satisfying (III), is also congruent to $M^c_{m,q}(a,b)$. For this paper we are now going to deal with these problems in a generic submanifold of CP^m. Generic submanifold $\pi(S^{p_1}(r_1) \times \cdots \times S^{p_N}(r_N))$ naturally satisfy the projected quantities (I), (II), (III), and (IV), because $S^{p_1}(r_1) \times \cdots \times S^{p_N}(r_N)$ mentioned above can be considered as a locally symmetric submanifold of S^{2n+1}. In this point of view, the converse problems such that the generic submanifold M of CP^m satisfying somewhat certain conditions of (I), (II), (III) or (IV) may be congruent to $\pi(S^{p_1}(r_1)\times\cdots\times S^{p_N}(r_N))$ will be occured.

§ 3. Generic submanifold satisfying certain conditions.

First we now let M be a generic submanifold of CP^m satisfying (I) and (IV). Then transvecting f_d^c to (IV) and using (1.7) gives that

$$(3.1)\qquad A_{ab}{}^x f_y{}^b = P_{yz}{}^x f_a{}^z,$$

where we have put

$$(3.2)\qquad P_{yz}{}^x = A_{eb}{}^x f_y{}^b f_z^e.$$

From which and (1.10), we see that $P_{yzx} = P_{yz}{}^w g_{wx}$ is symmetric for all indices. On the other hand, using (1.11) to (I), then we get

$$(3.3)\qquad (f_c^e A_{dex} + f_d^e A_{cex}) A_{cb}{}^x + (f_b^e A_{cex} + f_c^e A_{ebx}) A_{da}{}^x$$
$$- (f_b^e A_{dex} + f_d^e A_{ebx}) A_{ca}{}^x - (f_a^e A_{cex} + f_c^e A_{eax}) A_{db}{}^x = 0,$$

from which, transvecting f_z^a and using (IV) and (3.1), we find

$$P_{yz}{}^x f_d{}^y (A_{cex} f_b^e + f_c^e A_{bex}) - P_{yz}{}^x f_c{}^y (A_{dex} f_b^e + A_{ebx} f_d^e) = 0,$$

from which, also transvecting f_w^c and using (IV), (1.7) and (3.1), we get

$$(3.4)\qquad P_{wz}{}^x (f_b^e A_{dex} + f_d^e A_{bex}) = 0.$$

Now in this seciton we assume that generic submanifold M of CP^m has flat normal connection, then (1.13) implies that

$$f_b{}^x f_{ay} - f_a{}^x f_{by} + A_{be}{}^x A_a{}^e{}_y - A_{ae}{}^x A_b{}^e{}_y = 0,$$

from which, transvecting $f_z{}^a f_v{}^b$ and taking account of (3.1), we then have

$$(3.5)\qquad \delta_v^x g_{yz} - \delta_z^x g_{yv} = P_{zu}{}^x P_{vy}{}^u - P_{zy}{}^u P_{uv}{}^x.$$

Thus, from (3.4) and (3.5) we have the following

$$(f_a{}^e A_{dex} + f_d^e A_{aex}) g_{yz} - (f_a^e A_{dez} + f_d^e A_{aez}) g_{yx} = 0.$$

Contracting the above equation with respect to y and z, we then have that the induced f-structure on M is normal, that is, $f_a^e A_{dex} + f_d^e A_{aex} = 0$ for codimension $p>1$. When $p=1$, Maeda [4] proved those implications by using $P.C$ vector and principal curvature for the real hypersurface M of CP^m. Hence we get following by using Theorem A.

Theorem 3.1. *Let M be an n-dimensional complete generic submanifold of a complex projective space CP^m with flat normal connection. If M satisfies (I) and (IV) and if the mean curvature vector is parallel in the normal bundle, then*

M is of the form

$$\tilde{\pi}(S^{p_1}(r_1)\times\cdots\times S^{p_N}(r_N)),\quad p_1,\cdots\cdots,p_N \text{ are odd numbers} \geqq 1.$$

Next, if we let M be a generic submanifold satisfying (Ⅲ), then the certain generic submanifold with this condition will be determined.

Making use of (1.9) to the right side of (Ⅲ), then (Ⅲ) may be rewritten as follows

$$(3.6)\qquad (A_c{}^a{}_x f_{ea} + A_e{}^a{}_x f_{ca}) A_{dbx} - (A_d{}^a{}_x f_{ea} + A_e{}^a{}_x f_{da})$$
$$= (\nabla_e A_{dbx}) f_c{}^x - (\nabla_e A_{cbx}) f_d{}^x + f_{ce} f_{dx} f_b{}^x - f_{de} f_{cx} f_b{}^x,$$

Transvecting $f_y{}^e$ to (3.6) and making use of (1.7), we have

$$(f_c{}^a A_{ae}{}^x f_y{}^e) A_{dbx} - (f_d{}^a A_{ae}{}^x f_y{}^e) A_{cbx} = f_y{}^e (\nabla_e A_{dbx}) f_c{}^x - f_y{}^e (\nabla_e A_{cbx}) f_b{}^x,$$

from which, transvecting f^{db} gives that

$$(3.7)\qquad A_{cb}{}^x A_e{}^b{}_x f_z{}^e = P_z{}^{yx} A_{cbx} f_y{}^b,$$

where $P_z{}^{yx} = P_{zuv} g^{uy} g^{vx}$, and we have used (1.7) and skew-symmetry of f^{db}, symmetry of $A_{db}{}^x$. Thus, if we transvect $f_w{}^c$ to (3.7), we find

$$(A_{cb}{}^x f_w{}^c)(A_e{}^b{}_x f_z{}^e) = P_{wyx} P_z{}^{yx}.$$

Hence taking account of this equation and using (1.7), (3.2), then we get

$$\| A_{cb}{}^x f_w{}^c - P_{wu}{}^x f_b{}^u \|^2 = 0.$$

Thus, it follows that $A_{cb}{}^x f_w{}^c = P_{wu}{}^x f_b{}^u$, *i. e.*, (3.1) holds.

On the other hand, taking skew-symmetry e and b, and using equation of Codazzi (1.12), we may get

$$(3.8)\qquad (A_c{}^a{}_x f_{ea} + A_e{}^a{}_x f_{ca}) A_{db}{}^x - (A_c{}^a{}_x f_{ba} + A_c{}^a{}_x f_{ca}) A_{de}{}^x$$
$$- (A_d{}^a{}_x f_{ea} + A_e{}^a{}_x f_{da}) A_{cb}{}^x - (A_d{}^a{}_x f_{ba} + A_b{}^a{}_x f_{da}) A_{ce}{}^x = 0.$$

From which, transvecting $f_z{}^a f_w{}^b$, and noticing (3.1), (3.2), we get

$$P_{zwx}(A_c{}^a{}_x f_{ea} + A_e{}^a{}_x f_{ca}) = 0.$$

Therefore, we conclude that the structure tensor of M induced from CP^m is normal for $p>1$, since we have assumed that the normal connection of M is flat. For the real hypersurface of this case, Pak and present author proved these implications in the paper [7]. Combining these facts and Theorem A, we also get

Theorem 3.2. *Let M be an n-dimensional complete generic submanifold of a complex projective space CP^m with flat normal connection. If M satisfies* (Ⅲ) *and if the mean curvature vector is parallel in the normal bundle, then M is of the form*

$$\tilde{\pi}(S^{p_1}(r_1)\times\cdots\times S^{p_N}(r_N)),\quad p_1,\cdots,p_N \text{ are odd numbers} \geqq 1,$$
$$p_1+p_2+\cdots+p_N = n+1,\quad r_1^2+r_2^2+\cdots+r_N^2 = 1,\quad N = 2m-n+1,$$

where $S^{p_i}(r_i)$ *is a* p_i*-dimensional sphere with radius* r_i.

§ 4. Some pinching problems concerned with parallel Ricci tensor.

Let's denote by $\tilde{K}_{\beta\gamma}$ the components of the Ricci tensor of $\tilde{\pi}^{-1}(M)$, where M is a generic submanifold of CP^n. In this section we assume Ricci tensor is parallel along $\tilde{\pi}^{-1}(M)$, *i. e.*, $\nabla_\varepsilon \tilde{K}_{\beta\gamma}=0$. Now if we let $\tilde{K}_{cb}$ be the horizontal local function on $\{\tilde{U} : x^\circ\}$, then it is constant along each fibre $F\cap\tilde{U}$ and can be expressed as such (See [1])

$$\tilde{K}_{cb}=\tilde{K}_{\gamma\beta}E^\gamma{}_c E^\beta{}_b. \tag{4.1}$$

If we apply the operator $E^\varepsilon_e\nabla_\varepsilon=\nabla_e$ to (4.1) and make use of (2.3), we get

$$\nabla_e\tilde{K}_{cb}=f_{ce}\tilde{K}_{0b}+f_{be}\tilde{K}_{0c} \tag{V}$$

where $\tilde{K}_{0b}$ means that $\tilde{K}_{0b}=\tilde{K}_{\gamma\beta}E^\alpha{}_b\xi^\gamma$.

On the other hand, from (2.4) and (2.6) we find that

$$\tilde{K}_{cb}=K_{cb}+2f_c{}^e f_{eb}, \qquad \tilde{K}_{c0}=\nabla_e f_c{}^e, \tag{4.2}$$

where the Ricci tensor K_{cb} of M is given by

$$K_{cb}=(n+2)g_{cb}-3f_c{}^x f_{bx}+A_x A_{cb}{}^x-A_{cex}A_b{}^{ex}. \tag{4.3}$$

Then, substituting these equations into (V) and using (1.10), it follows

$$\nabla_e(-f_c{}^x f_{xb}+A_x A_{cb}{}^x-A_c{}^a{}_x A_{ba}{}^x)=(A_y-P_y)(f_{ce}f_b{}^y+f_{be}f_c{}^y) \tag{4.4}$$

where we have put $A_y=g^{ab}A_{aby}$ which is the mean curvature vector of M and have put $P_y=P_{xzy}g^{xz}$

Since it is natural that locally symmetric space should have parallel Ricci tensor, we will discuss the converse problem which is related to projected quantities (II) and (V). In this section we now suppose that M has parallel mean curvature vector. Then using (1.10) to (4.4), from which, applying $f_z{}^b$ and making use of (1.7) gives that

$$\begin{aligned} A_{edz}f_c{}^d+A_x(\nabla_e A_{cb}{}^x)f_z{}^b-(\nabla_e A_c{}^a{}_x)A_{ba}{}^x f_z{}^b-A_c{}^a{}_x(\nabla_e A_{ba}{}^x)f_z{}^b \\ =(A_z-P_z)f_{ce} \end{aligned} \tag{4.5}$$

On the other hand, contracting a and d to the equation (II) gives

$$\begin{aligned} (f_{ed}f_b{}^x+f_{eb}f_d{}^x+\nabla_e A_{db}{}^x)A_c{}^d{}_x+(f_{ec}f_d{}^x+f_{ed}f_c{}^x+\nabla_e A_{cd}{}^x)A_b{}^d{}_x \\ -A_x(f_{ec}f_b{}^x+f_{eb}f_c{}^x+\nabla_e A_{cb}{}^x)=0. \end{aligned} \tag{4.6}$$

Transvecting $f_z{}^b$ to (4.6) and using (1.7), then we have

$$\begin{aligned} A_c{}^d{}_z f_{ed}+(\nabla_e A_{db}{}^x)A_c{}^d{}_x f_z{}^b+(f_{ec}f_d{}^x+f_{ed}f_c{}^x+\nabla_e A_{cd}{}^x)A_b{}^d{}_x f_z{}^b \\ -A_z f_{ec}-A_x(\nabla_e A_{cb}{}^x)f_z{}^b=0. \end{aligned} \tag{4.7}$$

Combining (4.5) and (4.7) implies that

(4.8) $(A_{edz}f_c^{\ d}+A_{cdz}f_e^{\ d})+(f_{ed}A_b^{\ d}{}_x f_z^{\ b})f_c^{\ x}=0$.

from which, taking skew-symmetric part e and c, then we get

$$(f_{ed}A_b^{\ d}{}_x f_z^{\ b})f_c^{\ x}-(f_{cd}A_b^{\ d}{}_x f_z^{\ b})f_e^{\ x}=0 .$$

Thus transvecting $f_w^{\ c}$ to this equation, we easily get $f_{ed}A_b^{\ d}{}_x f_z^{\ b}=0$.
Hence (4.8) reduces to $A_{edz}f_c^{\ d}+A_{cdz}f_e^{\ d}=0$ which shows that the structure tensor is normal. Consequently, by using Theorem A it follows that

Theorem 4.1. *Let M be an n-dimensional complete generic submanifold of complex projective space CP^m with flat normal connection. If M satisfies* (II) *and* (IV), *and if the mean curvature vector is parallel in the normal bundle, then M is of the form as stated in Theorem* 3.2.

As a final remark, we are going to show that there dose not exist generic submanifold with parallel Ricci tensor in the class of $\pi(S^{p_1}(r_1)\times\cdots\times S^{p_N}(r_N))$. Thus we suppose that there exists a generic submanifold of $\pi(S^{p_1}(r_1)\times\cdots\times S^{p_N}(r_N))$ with parallel Ricci tenor. Furthermore, its mean curvature vector is parallel in the normal bundle, because $\nabla_c A_{ba}^{\ \ x}=-f_{ca}f_b^{\ x}-f_{cb}f_a^{\ x}$ is well known equivalent condition to the normality of the structure tensor f_{cb} (See [6]) for the generic submanifold of CP^m. Thus if we take covariant derivative to (4.3), then we get

$$3A_{da}^{\ \ y}f_c^{\ a}f_{by}+3A_{bay}f_b^{\ a}f_c^{\ y}-(f_{db}f_c^{\ y}+f_{dc}f_b^{\ y})A_y+(f_{da}f_{cy}+f_{ac}f_{ay})A_b^{\ ay}$$
$$+(f_d^{\ a}f_b^{\ y}+f_{db}f^{ay})A_{cay}=0 .$$

from which, transvecting $f_z^{\ c}$ and using (1.7), normality of f, then we have

$$3A_{dax}f_b^{\ a}-A_xf_{db}+f_{da}A_b^{\ a}{}_x+P_xf_{db}=0 .$$

From which, also using normality of f, we may find

(4.9) $$2A_{dax}f_b^{\ a}+(A_x-P_x)f_{bd}=0 .$$

Applying f^{bd} to (4.9) and making use of (1.7) and (3.2) imply $A_x=P_x$.
Hence using this fact, (4.9) reduces to

(4.10) $$A_{dax}f_b^{\ a}=0 .$$

Taking covariant derivative to (4.10) and using (1,9), we may get

$$(\nabla_e A_{dax})f_b^{\ a}+A_{dax}(A_e^{\ a}{}_y f_b^{\ y}-A_{eb}^{\ \ y}f_y^{\ a})=0 ,$$

from which, transvecting $f_c^{\ b}$ and using (4.10) imply

(4.11) $$\nabla_e A_{dc}^{\ \ x}=(f_y^{\ a}\nabla_e A_{da}^{\ \ x})f_c^{\ y},$$

from which, transvecting with $f_w^{\ c}$,

$$f_w^{\ c}\nabla A_{dc}^{\ \ x}=-f_{ed}\delta_w^{\ x},$$

which and (4.11) give $f_{ec}f_d^{\ z}=0$. Hence it follows that f_{cb} identically vanishes.

This is a contradiction. Thus from above facts we have

Theorem 4.2. *There does not exist a generic submanifold of* CP^n *with parallel Ricci tensor in the class of* $\pi(S^{p_1}(r_1)\times\cdots\times S^{p_N}(r_N))$.

Classically it is well known that Einstein generic submanifold has parall Ricci tensor. Thus from Theorem 4.2. we also get

Coroslary 4.3. *There does not exist an Einstein generic sub manifold of* CP^n *in the class of* $\pi(S^{p_1}(r_1)\times\cdots\times S^{p_N}(r_N))$.

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