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A Submanifold of a Complex Space Form

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§ 0. Introduction

In 1978, K. Yano and U-Hang Ki [7] studied the so-called $(f, g, u, v, w, \lambda, u, \nu)$-structure induced on a submanifold of codimension 3 immersed in an almost Hermitian manifold and found some conditions to admit the almost contact metric structure on the submanifold. In 1980, Y. Tashiro and I.-B. Kim [5] generalized the notion of $(f, g, u, v, w, \lambda, \mu, \nu)$-structure, which is called the metric compound structure.

On the other hand, the present authors [6] examined submanifolds of a Kaehlerian manifold of constant holomorphic sectional curvature with the so-called almost contact metric compound structure in the sense that the submanifolds admit the almost contact metric structure.

The purpose of the present paper is to characterize submanifolds with the almost contact metric compound structure immersed in an even-dimensional Euclidean space under some conditions.

We shall use the following theorem in order to examine the properties of submanifolds admitting an almost contact metric compound structure immersed in a Kaehlerian manifold of constant holomorphic sectional curvature c.

Theorem A ([5]). *Let M be a submanifold of codimension l with the induced almost contact metric structure (f, g, v, f^{λ}) of an even-dimensional Euclidean space E^n.*

If the submanifold M satisfies one of the followings;

(1) *M of dimension $m>3$ is umblical with respect to the distinguished normal N^{λ} and N^{λ} parallel to the mean curvature vector,*

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(2) *M of dimension $m>3$ is pseudo-umblical submanifold and the distinguished normal N^λ parallel to the mean curvature vector,*

(3) *The distinguished normal N^λ is concurrent,*

then M is the intersection of a complex cone with generator N^λ and an $(n-1)$-dimensional sphere.

Manifolds, submanifolds and other geometric objects appeared in this paper are assumed to be differentiable and of C^∞. We use throughout this paper the systems of indices as follows :

$A, B, C, \cdots=1, 2, \cdots, 2m \quad : \quad h, i, j, \cdots=1, 2, \cdots, n$

$u, v, w, x, y, z=1^*, 2^*, \cdots, p^*, \quad n+p=2n$

$(u), (v), (w), (x), (y), (z)=2^*, \cdots, p^*.$

The summation convention will be used with respect to these systems of indices.

§ 1. Preliminaries

Let $\widetilde{M}$ be a $2m$-dimensional almost Hermitian manifold covered by a system of coordinate neighborhoods $\{\widetilde{U}; x^A\}$ and (F, G) the almost Hermitian structure, where F is the almost Hermitian metrc tensor of M. We denotes by F_B^A and G_{CB} components of F and G respectively. Then we have

$$F_B^A F_C^B=-\delta_C^A, \qquad F_C^D F_B^E G_{DE}=G_{CB}, \tag{1.1}$$

δ_C^A being the Kronecker delta.

If we put the covariant components of F as

$$F_{CB}=F_C^A G_{BA}, \tag{1.2}$$

then F_{CB} is skew-symmetric with respect to the indices C and B.

Let M be an n-dimensional Riemannian manifold covered by a system of coordinate neighborhoods $\{U: x^h\}$ and immersed isometrically in $\widetilde{M}$ by the immersion $i: M\rightarrow \widetilde{M}$. We idenfy $i(M)$ with M itself and represent the immersion locally by

$$x^A=x^A(x^h). \tag{1.3}$$

We now put $B_i^A=\partial_i x^A (\partial_i=\partial/\partial x^i)$. Then B_i^A are n linearly independent vectors of $\widetilde{M}$ tangent to M. And denote by C_x^A mutually orthegonal unit normal vector fields of M. Then we have $G_{CB}B_i^B C_x^C=0$ and the metric tensor of the normal bundle of M is given by $g_{zy}=G_{CB}C_z^C C_y^B=\delta_{zy}$. Therefore, vector fields B_i^A and C_x^A span the tangent space $T_P(M)$ of M at every point P of M. The metric tensor g of M induced from that of $\widetilde{M}$ is given by

(1.4) $$g_{ji}=G_{CB}B_j^C B_i^B$$

since the immersion is isometric.

The transforms of the tangent vectors B_j^A and the normal vectors C_x^A to M by F are expressed in the form

(1.5) $$F_B^A B_j^B=f_j^h B_h^A+f_j^x C_x^A,$$

(1.6) $$F_B^A C_x^B=-f_x^h B_h^A+f_x^y C_y^A,$$

where f_j^h are components of a tensor field of type (1, 1), f_j^x those of 1-form for each fixed x, f_x^h vector field associated with f_j^x given by $f_x^h=f_j^y g^{jh} g_{yx}$, f_x^y function for fixed indices x and y. Putting $f_{ji}=f_j^h g^{hi}$, $f_{jx}=f_j^y g_{yx}$, $f_{xj}=f_x^h g_{hj}$ and $f_{xy}=f_x^z g_{zy}$, we can easily find

(1.7) $$f_{ji}=-f_{ij},\quad f_{jx}=f_{xj},\quad f_{xy}=-f_{yx}.$$

Applying F to (1.5) and (1.6) respectively and using (1.1) and those expressions, we have

(1.8) $$f_j^t f_t^h=-\delta_j^h+f_j^x f_x^h,$$

(1.9) $$f_j^t f_t^y+f_j^x f_x^y=0,\quad f_x^t f_t^i+f_x^y f_y^i=0,$$

(1.10) $$f_y^z f_z^x=-\delta_y^x+f_y^t f_t^x.$$

The second equation of (1.1) and (1.4) imply

(1.11) $$f_j^t f_i^s g_{ts}=g_{ji}-f_j^x f_{ix}.$$

Now, removing the almost Hermitian ambient manifold $\widetilde{M}$, we suppose that an n-dimensional Riemannian manifold M admits a metric tensor g_{ji}, a tensor field f_j^h of type (1, 1), p vector fields f_x^h, p 1-forms f_j^x and $p(p-1)/2$ scalar fields f_{xy} satisfying the relationships (1.8)~(1.11). Such a set $(f_j^h, g_{ji}, f_x^h, f_x^y)$ is said to be a ***metric compound structure*** on M.

If we put

(1.12) $$F=\begin{pmatrix} f_i^h & -f_x^h \\ f_{xi} & f_{xy} \end{pmatrix} \quad \text{and} \quad G=\begin{pmatrix} g_{ji} & 0 \\ 0 & \delta_{yx} \end{pmatrix}$$

then the set (F, G) defines an almost Hermitian structure in the product manifold $M\times R^p$ of the manifold M with a p-dimensional Euclidean space R^p.

We suppose that M admits an almost contact metric compound structure. Then we have

(1.13) $$f_j^t f_t^h=-\delta_j^h+p_j p^h,$$

(1. 14) $$f_j^t p_t = 0, \quad f_j^h p^j = 0,$$

(1. 15) $$p_i p^i = 1$$

and

(1. 16) $$f_j^t f_i^s g_{ts} = g_{ji} - p_j p_i,$$

where p_j is a 1-form and p^h vector field associated with p_j given by $p^h = g^{hj} p_j$ on M.

In this case we know that the dimension n of M is odd and the rank (f_j^i) is equal to n-1.

Comparing (1. 11) and (1. 16), we have

(1. 17) $$f_j^x f_{ix} = p_j p_i.$$

This equation shows that the product of the matrix (f_j^x) with its transpose is of rank 1 and hence the matrix (f_j^x) by itself is of 1.

Therefore, we may put

(1. 18) $$f_j^x = \nu^x p_j,$$

where ν^x are certain scalar fields for each x.

Since $f_j^x f_x^j = p_j p^j = 1$, we have

(1. 19) $$\nu^x \nu_x = 1.$$

and hence (1. 9) and (1. 10) are reduce respectively to

(1. 20) $$f_x^y \nu^x = 0, \quad \nu_y f_x^y = 0$$

and

(1. 21) $$f_y^z f_z^x = -\delta_y^x + \nu_y \nu^x.$$

The equations (1. 19) ~ (1. 21) form an almost contact metric structure on R^p at every point of M, and consequently we see that the dimension p of R^p is odd.

Conversely, assuming that an almost contact metric structure $(f_y^x,\ g_{yx},\ \nu^x)$on R^p is admitted, we can prove that the metric compound structure $(f_j^h,\ g_{ji},\ f_x^h,\ f_y^x)$ induces an almost contact metric structure $(f_j^h,\ g_{ji},\ p^h)$ on M.

Thus we have

Theorem B ([5]). *Let* $(f_j^h,\ g_{ji},\ f_x^h,\ f_y^x)$ *be a metric compound structure on* M. *In order that* f_j^h *and* g_{ji} *constitute an almost contact metric structure* $(f_j^h,\ g_{ji},\ p^h)$ *on* M, *it is necessary and sufficient that* f_y^x *and* g_{yx} *constitute an almost contact structure* $(f_y^x,\ g_{yx},\ \nu^x)$ *on* R^p *at every point of* M.

From above discussions, we also have

Theorem C ([5]) *In order for a metric compound structure* $(f_j{}^h, g_{ji}, f_x{}^h, f_y{}^x)$ *to be almost contact metric structure, it is necessary and sufficient that the matrix* $(f_x{}^h)$ *is of rank* 1, *that is, the* p *vector fields* $f_x{}^h$ *are all parallel to one another.*

A metric compound structure admitting an almost contact metric structure is said to be an ***almost contact metric compound structure*** on M

A metric compound structure is said to be ***antinormal*** if the tensor field $S_{ji}{}^h$ of type (1,2) defined by

$$S_{ji}{}^h = N_{ji}{}^h + (\partial_j f_i{}^x - \partial_i f_j{}^x) f_x{}^h \tag{1.22}$$

satisfies

$$S_{ji}{}^h = 2(f_j{}^x \partial_i f_x{}^h - f_i{}^x \partial_j f_x{}^h), \tag{1.23}$$

where $N_{ji}{}^h$ is the Nijenhuis tensor formed with $f_j{}^h$, that is,

$$N_{ji}{}^h = f_j{}^t \partial_t f_i{}^h - f_i{}^t \partial_t f_j{}^h - (\partial_j f_i{}^t - \partial_i f_j{}^t) f_t{}^h.$$

§ 2. Structure equations induced on a submanifold of an almost Hermitian manifold

In this section we assume that M is an n-dimensional submanifold of codimension p of an almost Hermitian manifold and admits an almost contact metric compound structure $(f_j{}^h, g_{ji}, f_x{}^h, f_y{}^x)$ and consequently $(f_j{}^h, g_{ji}, p^h)$ defines an almost contact metric structure as shown in § 1. The vector field N^A defined by

$$N^A = \nu^x C_x{}^A \tag{2.1}$$

is unit normal to M in $\widetilde{M}$.

As to transforms of the tangent vectors $B_i{}^A$ and the normal vector N^A by the almost complex structure tensor $\widetilde{F}$, we have

$$\widetilde{F}_B{}^A B_j{}^B = f_j{}^t B_t{}^A + p_j N^A, \tag{2.2}$$

$$\widetilde{F}_B{}^A N^B = -p^t B_t{}^A \tag{2.3}$$

respectively because of (1.21), (1.23), (1.24) and (2.1).

It is well known that the submanifold M of an almost Hermitian manifold $\widetilde{M}$ satisfying (2.2) is so-called semi-invariant with respect to N^A and N^A is said to be the distinguished normal to M (cf. [4]).

We now take the distinguished normal N^A as C^A. Then we have from (2.1) $\nu^{1*}=1$ and $\nu^x=0$. And consequently we find from (1.21), (1.23) and (1.24)

(2.4) $$f_j^{1*}=p_j, \qquad f_{1*}^h=p^h,$$

(2.5) $$f_y^{1*}=0,$$

(2.6) $$f_j^{(x)}=0,$$

(2.7) $$f_{(x)}^{(y)}f_{(y)}^{(z)}=-\delta_{(x)}^{(z)}.$$

Therefore, (1.9) and (1.10) respectively reduce to

(2.8) $$\tilde{F}_B^A B_j^B=f_j^t B_t^A+p_j C^A,$$

(2.9) $$\tilde{F}_B^A C^B=-p^t B_t^A,$$

(2.10) $$\tilde{F}_B^A C_{(x)}^B=f_{(x)}^{(y)} C_{(y)}^A$$

with the aid of (2.4), (2.5) and (2.6).

Denoting by ∇_j the operator of van der Waerden-Bortolotti covariant differentiation with respect to g_{ji}, we have the equations of Gauss for M in $\tilde{M}$

(2.11) $$\nabla_j B_i^A=A_{ji}C^A+A_{ji}^{(x)}C_{(x)}^A,$$

where A_{ji} and A_{ji}^{x} are the second fundamental tensors with respect to C^A and $C_{(x)}^A$ respectively, and those of Weingarten

(2.12) $$\nabla_j C^A=-A_j^t B_t^A+l_j^{x} C_{(x)}^A,$$

(2.13) $$\nabla_j C_{(x)}^A=-A_{j\ (x)}^{t} B_t^A-l_{j(x)} C^A+l_{j(x)}^{(y)} C_{(y)}^A,$$

where $A_j^h=g^{ht}A_{jt}$, $A_{j\ (x)}^{h}=A_{ji}^{(y)}g_{(y)(x)}g^{hi}=A_{ji(x)}g^{hi}$, l_j^{x} and $l_{i(x)}^{(y)}$ are the third fundamental tensors, $l_{j(x)}$ being given by $l_{j(x)}=l_j^{(y)}g_{(y)(x)}$.

Putting $l_{j(x)(y)}=l_{j(x)}^{(z)}g_{(z)(y)}$, we can easily find

$$l_{j(x)(y)}=-l_{j(x)(x)}.$$

We now assume that the ambient manifold $\tilde{M}$ is a Kaehlerian manifold, that is, $\nabla\tilde{F}=0$.

Differentiating (2.8), (2,9) and (2.10) covariantly respectively, and using (2.11), (2.12) and (2.13), we find

(2.14) $$\nabla_j f_i^h=-A_{ji}p^h+A_j^h p_i,$$

(2.15) $$\nabla_j p_i=-A_{jt}f_i^t, \qquad \nabla_j p^h=A_j^t f_t^h,$$

(2.16) $$A_{ji}^{(x)}f_{(x)}^{(y)}=A_{jt}^{(y)}f_i^t+l_j^{(y)}p_i,$$

(2.17) $$A_{jt}^{(x)}p^t=-l_j^{(y)}f_{(y)}^{x},$$

(2.18) $$\nabla_j f_{(y)}^{(x)}=l_{j(y)}^{(z)}f_{(z)}^{(x)}-f_{(y)}^{(z)}l_{j(z)}^{(x)}.$$

Transvecting (2.16) with $f_{(y)}^{z}$ and taking account of (2.7), we get

(2. 19) $$A_{ji}{}^{(z)} = -A_{jt}{}^{(y)} f_{(y)}{}^{(z)} f_i{}^t - l_j{}^{(y)} f_{(y)}{}^{(z)} p_i,$$

from which,

(2. 20) $$A^{(z)} = -p^t l_t{}^{(y)} f_{(y)}{}^{(z)},$$

where we have put $A^{(z)} = g^{ji} A_{ji}{}^{(z)}$.

If we transvect (2. 19) with $f_h{}^j$, then we find

$$A_{ji}{}^{(z)} f_h{}^j = -A_{jt}{}^{(y)} f_{(y)}{}^{(z)} f_h{}^j f_i{}^t - l_j{}^{(y)} f_h{}^j f_{(y)}{}^{(z)} p_i,$$

or, use (2. 16),

$$A_{ih}{}^{(x)} f_{(x)}{}^{(z)} - l_i{}^{(z)} p_h = -A_{jt}{}^{(y)} f_{(y)}{}^{(z)} f_h{}^j f_i{}^t - l_j{}^{(y)} f_h{}^j f_{(y)}{}^{(z)} p_i.$$

Taking the skew-symmetric part with respect to the indices i and h, we obtain

$$-l_i{}^{(z)} p_h + l_h{}^{(z)} p_i = -l_j{}^{(y)} f_{(y)}{}^{(z)} (f_h{}^j p_i - f_i{}^j p_h),$$

from which, transvecting p^i,

(2. 21) $$f_h{}^t l_t{}^{(y)} = A^{(y)} p_h + l_h{}^{(x)} f_{(x)}{}^{(y)}$$

because of (2. 7) and (2. 20). Transvection $l^{h(x)}$ yields

$$A^{(x)} A^{(z)} f_{(z)}{}^{(y)} + l_t{}^{(z)} l^{t(y)} f_{(z)}{}^{(x)} + A^{(y)} A^{(z)} f_{(z)}{}^{(x)} + l_t{}^{(z)} l^{t(x)} f_{(z)}{}^{(y)} = 0$$

with the aid of (2. 20). Also, transvecting $f_{(x)}{}^{(w)}$ and using (2. 7), we find

(2. 22) $$(A^{(y)} A^{(w)} + l_t{}^{(y)} l^{t(w)}) - (A^{(x)} A^{(z)} + l_t{}^{(z)} l^{t(x)}) f_{(z)}{}^{(y)} f_{(x)}{}^{(w)} = 0 .$$

Now we assume that the almost contact metric compound structure $(f_j{}^i, g_{ji}, f_i{}^x, f_y{}^x)$ is antinormal. Then (1. 23) reduces to

$$f_j{}^t \nabla_t f_i{}^h - f_i{}^t \nabla_t f_j{}^h - (\nabla_j f_i{}^t - \nabla_i f_j{}^t) f_t{}^h + (\nabla_j p_i - \nabla_i p_j) p^h$$
$$= 2 p_j \nabla_i p^h - 2 p_i \nabla_j p^h$$

because of (2. 4) ~ (2. 6). Substituting (2. 14) and (2. 15) into this, we obtain

$$(A_j{}^t f_t{}^h + f_j{}^t A_t{}^h) p_i - (A_i{}^t f_t{}^h + f_i{}^t A_t{}^h) p_j = 0$$

with the aid of (1. 17), or, equivalently,

$$(A_{ht} f_i{}^t - A_{jt} f_h{}^t) p_i - (A_{ht} f_i{}^t - A_{it} f_h{}^t) p_j = 0 .$$

Transvection this with p^i gives

$$A_{ht} f_j{}^t - A_{jt} f_h{}^t = q_h p_j$$

for some covector q_h is given by $q_h = -(A_{it} p^i) f_h^t$.

Taking the symmetric part of this with respect to the indices h and j, we find

$$q_h p_j + q_j p_h = 0 .$$

Transvecting p^j to this, we see that $q_h=0$ by its definition. Consequently we have

$$A_{jt}f_i^t - A_{it}f_j^t = 0. \tag{2.23}$$

Thus, we get

Theorem 2.1. *Let M be a semi-invariant submanifold with the distinguished normal C^A admitting an almost contact metric compound structure immersed in a Kaehlerian manifold $\widetilde{M}$. Then in order for this structure to be antinormal, it is necessary and sufficient that the second fundamental tensor A with respect to the distinguished normal and the structure tensor f anticommute.*

The equations of Gauss for M in a Kaehlerian manifold $\widetilde{M}$ are given by

$$K_{kji}{}^h = K_{DCB}{}^A B_k{}^D B_j{}^C B_i{}^B B^h{}_A + A_k{}^h A_{ji} - A_j{}^h A_{ki} + A_k{}^h{}_{(x)} A_{ji}{}^{(x)} - A_j{}^h{}_{(x)} A_{ki}{}^{(x)}, \tag{2.24}$$

where $K_{DCB}{}^A$ and $K_{kji}{}^h$ are the Riemann-Christoffel curvature tensors of $\widetilde{M}$ and M respectively, and we have put $B^h{}_A = B_i{}^B g^{hi} \widetilde{g}_{AB}$.

We now assume that the ambient manifold $\widetilde{M}$ is a Kaehlerian manifold of constant holomorphic sectional curvature c, that is, its curvature tensor has the form

$$K_{DCB}{}^A = \frac{c}{4}(\delta_D^A g_{CB} - \delta_C^A g_{DB} + \widetilde{F}_D{}^A \widetilde{F}_{CB} - \widetilde{F}_C{}^A \widetilde{F}_{DB} - 2\widetilde{F}_{DC}\widetilde{F}_B{}^A).$$

Substituting this into (2.24) and using (1.4) and (2.8), we find

$$\begin{aligned} K_{kji}{}^h = {} & \frac{c}{4}(\delta_k^h g_{ji} - \delta_j^h g_{ki} + f_k{}^h f_{ji} - f_j{}^h f_{ki} - 2f_{kj}f_i{}^h) + A_k{}^h A_{ji} \\ & - A_j{}^h A_{ki} + A_k{}^h{}_{(x)} A_{ji}{}^{(x)} - A_j{}^h{}_{(x)} A_{ki}{}^{(x)}. \end{aligned} \tag{2.25}$$

Taking account of (2.8), (2.9), (2.10), (2.12) and (2.13), we have also the equations of Mainardi-Codazzi :

$$\nabla_k A_{ji} - \nabla_j A_{ki} - l_{k(x)} A_{ji}{}^{(x)} + l_{j(x)} A_{ki}{}^{(x)} = \frac{c}{4}(p_k f_{ji} - p_j f_{ki} - 2p_i f_{kj}), \tag{2.26}$$

$$\nabla_k A_{ji}{}^{(x)} - \nabla_j A_{ki}{}^{(x)} + l_k{}^{(x)} A_{ji} - l_j{}^{(x)} A_{ki} + l_{k(y)}{}^{(x)} A_{ji}{}^{(y)} - l_{j(y)}{}^{(x)} A_{ki}{}^{(y)} = 0, \tag{2.27}$$

and those of Ricci are given by

$$\nabla_j l_i{}^{(x)} - \nabla_i l_j{}^{(x)} + A_j{}^t A_{ti}{}^{(x)} - A_i{}^t A_{jt}{}^{(x)} + l_{j(y)}{}^{(x)} l_i{}^{(y)} - l_{i(y)}{}^{(x)} l_j{}^{(y)} = 0, \tag{2.28}$$

$$\begin{aligned} & \nabla_j l_{i(x)}{}^{(y)} - \nabla_i l_{j(x)}{}^{(y)} + A_j{}^t{}_{(x)} A_{it}{}^{(y)} - A_i{}^t{}_{(x)} A_{jt}{}^{(y)} + l_{j(x)} l_i{}^{(y)} - l_{i(x)} l_j{}^{(y)} + l_{j(z)}{}^{(y)} l_{i(x)}{}^{(z)} \\ & - l_{i(z)}{}^{(y)} l_{j(x)}{}^{(z)} = \frac{1}{2} f_{ij} f_{(x)}{}^{(y)}. \end{aligned} \tag{2.29}$$

§ 3. Some characterizations of submanifolds of a Kaehlerian manifold of constant holomorphic sectional curvature c admitting an almost contact metric compound structure

In this section we assume that the metric compound structure induced on n-dimensional submanifold M of a Kaehlerian manifold $\widetilde{M}$ of constant holomorphic sectional curvature c defines an almost contact metric structure. Consequently (f_j^h, g_{ji}, p^h) defines an almost contact metric structure on M.

We now suppose that the submanifold M is umbilical with respect to the distinguished normal C^A, that is,

$$A_{ji}=\rho g_{ji}, \qquad A^{(x)}=0, \tag{3.1}$$

for some scalar field ρ. Then (2. 15) becomes $\nabla_j p_i=\rho f_{ji}$, which implies that

$$\nabla_k \nabla_j p_i=(\nabla_k\rho) f_{ji}+\rho^2 (g_{ki}p_j-g_{ki}p_j)$$

with the aid of (2. 14) and (3. 1). Taking account of the Ricci identity

$$-K_{kji}{}^t p_t=(\nabla_k\rho) f_{ji}-(\nabla_j\rho) f_{ki}+\rho^2 (g_{ki}p_j-g_{ji}p_k),$$

from which, using the first Bianchi identity,

$$(\nabla_k\rho) f_{ji}+(\nabla_j\rho) f_{ik}+(\nabla_i\rho) f_{kj}=0.$$

From this we can see that ρ is a constant.

We now assume that the second fundamental tensor $A_{ji}{}^{x}$ with respect to the normal vectors $C_{(x)}{}^A$ and the structure tensor f_j^h commute, that is

$$A_j{}^t{}_{(x)} f_t^h - f_j^t A_t{}^h{}_{(x)}=0,$$

or, equivalently,

$$A_{jt}{}^{x} f_i^t + A_{it}{}^{x} f_j^t=0. \tag{3.2}$$

If we take the symmetric part of (2. 19) with respect to the indices j and i and make use of (3. 2), then we find

$$2A_{ji}{}^{z}=-(l_j{}^{y} p_i+l_i{}^{y} p_j) f_{(y)}{}^{z}.$$

Transvection this with p^i yields

$$l_j{}^{z}=(l_t{}^{z} p^t) p_j$$

because of (2. 7) and (2. 17). Since (2. 20) implies that $l_j{}^{z}=0$ because of (3. 1). And consequently

$$A_{ji}{}^{x}=0. \tag{3.3}$$

Thus, (2.26) reduces to

$$\frac{c}{4}(p_k f_{ji} - p_j f_{ki} - 2p_i f_{kj}) = 0,$$

which shows that $c=0$, that is, the ambient manifold $\tilde{M}$ is an Euclidean space. Combining these facts and Theorem A in § 0, we get

Theorem 3.1. *Let M be an n (>3)-dimensional submanifold admitting an almost contact metric compound structure immersed in an $2m$-dimensional Kaehlerian manifold $\tilde{M}$ of constant holomorphic sectional curvature c. If M is umbilical with respect to the distinguished normal C^A and the second fundamental tensor $A_{ji}{}^{(x)}$ with respect to the normal vector fields $C_{(x)}{}^A$ and the structure tensor $f_j{}^h$ commute, then M is an intersection of complex cone with generator C^A and a $(m-1)$-dimensional sphere.*

We now assume that the induced almost contact metric compound structure on M is antinormal. We then have from (2.23)

$$A_{jt} f_i{}^t - A_{it} f_j{}^t = 0. \tag{3.4}$$

Transvecting (3.4) with p^j and using (1.17), we find

$$(A_{jt} p^j) f_i{}^t = 0,$$

from which, transvecting $f_k{}^i$,

$$A_{jt} p^t = \alpha p_j$$

with the aid of (1.16), where α is some function given by $\alpha = A_{ji} p^j p^i$.
Transvection (3.4) with f^{ji} gives

$$A = \alpha \tag{3.5}$$

because of (1.16), where we have put $A = A_{ji} g^{ji}$.
Therefore, we obtain

$$A_{jt} p^t = A\, p_j. \tag{3.6}$$

Differentiating (3.5) covariantly and using (2.15), we find

$$(\nabla_k A_{jt}) p^t - A_j{}^t A_{ks} f_t{}^s = (\nabla_k A) p_j - A A_{kt} f_j{}^t,$$

from which, taking the skew-symmetric part in k and j,

$$\{ l_{k(x)} A_{jt}{}^{(x)} - l_{j(x)} A_{kt}{}^{(x)} + \frac{c}{4}(p_k f_{jt} - p_j f_{kt} - 2p_t f_{kj}) \} p^t - 2A_j{}^t A_{ks} f_t{}^s$$
$$= (\nabla_k A) p_j - (\nabla_j A) p_k$$

because of (2.25). Making use of (1.17), (1.18) and (2.17), we obtain

(3. 7) $$-\frac{c}{2}f_{kj}-2\,l_{k(x)}l_j^{(y)}f_{(y)}{}^{(x)}-2\,A_j{}^tA_{ks}f_t{}^s=(\nabla_kA)\,p_j-(\nabla_jA)\,p_k.$$

Transvecting this with p^j and using (3. 6), we see that

$$\nabla_kA=(p^t\nabla_tA)\,p_k+2\,l_{k(x)}A^{(x)}$$

because of (2. 20). Thus, (3. 7) reduces to

(3. 8) $$-\frac{c}{4}f_{kj}-l_{k(x)}l_j^{(y)}f_{(y)}{}^{(x)}=A_j{}^tA_{ts}f_k{}^s+A^{(x)}\,(l_{k(x)}p_j-l_{j(x)}p_k)$$

with the aid of (3. 4). Transvecting this with $f_h{}^k$ and using (1. 16) and (1, 17), we get

$$\frac{c}{4}\,(g_{hj}-p_hp_j)-l_{k(x)}l_{j(y)}f^{(y)(x)}f_h{}^k=A_j{}^tA_{ts}\,(-\delta_h{}^s+p_hp^s)$$
$$+A^{(x)}f_h{}^k\,l_{k(x)}p_j,$$

or, taking account of (2. 21),

(3. 9) $$A_j{}^tA_{ht}+\frac{c}{4}g_{jh}=(A_{(x)}A^{(x)}+A^2+\frac{c}{4})\,p_hp_j+l_{j(y)}f^{(y)(x)}A_{(x)}p_h+l_{j(x)}l_h{}^{(x)}$$
$$+A^{(x)}\,l_h{}^{(y)}f_{(y)(x)}p_j,$$

which implies

$$A_{ji}A^{ji}=\frac{1-n}{4}c+A^2-A_{(x)}A^{(x)}+l_{t(x)}l^{t(x)}$$

with the aid of (2. 20). Thus it becomes

(3. 10) $$\|A_{ji}-Ap_jp_i\|^2=-(A_{(x)}A^{(x)}+\frac{n-1}{4}c)+l_{t(x)}l^{t(x)}$$

because of (3. 6).

We now suppose that the distinguished normal vector C^A is parallel in the normal bundle, that is, $l_j{}^{(x)}=0$ and $c\geqq 0$. Then, we see from (3. 10) that $c=0$ and $A_{(x)}=0$. If the submanifold M is minimal, then we can see from (2. 25) ~ (2. 29) and (3. 10) that M is a submanifold of a $(2m-1)$-dimensional Euclidean space E^{2m-1}.

Hence we have

Theorem 3. 2. *Let M be an n-dimansional minimal submanifold admitting an antinormal almost contact metric compound structure of a $2m$-dimensional Kaehlerian manifold $\widetilde{M}$ of constant holomorphic sectional curvature $c\geqq 0$. If the distinguished normal C^A is parallel in the normal bundle, then M is a submanifold of a $(2m-1)$-dimensional Euclidean space E^{2m-1}.*

Now, we denote by $K_{ji}=K_{tji}{}^t$ and K the Ricci tensor and the scalar curvature of M respectively. Then we have from (2. 25)

$$K_{ji}=\frac{c}{4}\{(n+2)g_{ji}-3p_jp_i\}+AA_{ji}-A_j^tA_{it}+A_{(x)}A_{ji}{}^{(x)}-A_{j(x)}^tA_{it}{}^{(x)},$$

from which

$$K=\frac{c}{4}\{(n+2)n-3\}+A^2-A_{ji}A^{ji}+A_{(x)}A^{(x)}-A_{ji}{}^{(x)}A^{ji}{}_{(x)}.$$

Taking account of (2.19), (2.20) and (3.10), K becomes

(3.11) $$K=\frac{c}{4}(n-1)(n+4)-\|A_{ji}{}^{(x)}-A^{(x)}p_jp_i\|^2+(A_{(x)}A^{(x)}-l_t{}^{(x)}l^t{}_{(x)}).$$

On the other hand, the following relationship holds:

(3.12) $$\begin{aligned}&(l_{j(x)}p_i-l_{i(x)}p_j)(l^{j(x)}p^i-l^{i(x)}p^j)\\&=2\,l_{t(x)}l^{t(x)}+2\,A_{(y)}f_{(x)}{}^{(y)}A^{(w)}f_{(w)}{}^{(x)}\\&=2\,(l_{t(x)}l^{t(x)}-A_{(x)}A^{(x)})\end{aligned}$$

because of (2.7) and (2.20). Hence (3.11) reduces to

(3.13) $$K=\frac{c}{4}(n-1)(n+4)-\|A_{ji}{}^{(x)}-A^{(x)}p_jp_i\|^2-\frac{1}{2}\|l_{j(x)}p_i-l_{i(x)}p_j\|^2.$$

If $K\geqq\frac{c}{4}(n-1)(n+4)$, then we have

(3.14) $$A_{ji}{}^{(x)}=A^{(x)}p_jp_i,\qquad l_{j(x)}p_i-l_{i(x)}p_j=0.$$

We now suppose that the distinguished normal C^A is parallel in the normal bundle and $c\geqq 0$. Then we can see from (2.20), (3.10) and (3.14) that

(3.15) $$A_{ji}=Ap_jp_i,\qquad A_{ji}{}^{(x)}=0\ \text{ and }\ c=0.$$

Thus (2.12) and (2.13) respectively reduce to

$$\nabla_jC^A=-Ap_j(p^tB_h{}^A),\qquad \nabla_jC_{(x)}{}^A=l_{j(x)}{}^{(y)}C_{(y)}{}^A$$

because of $l_{j(x)}=0$. Since (2.15) becomes

(3.16) $$\nabla_jp^h=0$$

because of (3.15), a real hypersurface M_0 of M can be defined by the Pfaffian form $w=p_idx^i$. We assume that M_0 is covered by a system of coordinate neighborhoods $\{U_0:\xi^a\}$, where the indices a, b and c run over the range $\{1,2,\cdots,n-1\}$. Let

$\tilde{i}: M^0\to M$ be an isometric immersion represented by $x^h=x^h(\xi^a)$. Putting $B_a{}^h=\partial_a y^h\,(\partial_a=\partial/\partial\xi^a)$, then B_a^h are $2n$ linearly independent vectors of M tangent to M_0. We now put

(3.17) $$B_a{}^A=B_a{}^jB_j{}^A,\qquad p^A=p^tB_t{}^A.$$

Then p^A is a unit normal vector field orthogonal to C^A and $C_{(x)}{}^A$. In this case, we can see that M_0 is a totally geodesic submanifold of a Euclidean space E^{2m} of dimension $2m$ because of $c=0$. Consequently, M_0 is a $(n-1)$-dimensional plane E^{n-1} parallel along p^h. Thus M is a cylindrical surface or a plane.

Hence we have

Theorem 3. 3. *Let M be an n-dimensional submanifold admitting an antinormal almost contact metric compound structure of $2m$-dimensional Kaehlerian manifold $\tilde{M}$ of constant holomorphic sectional curvature $c \geqq 0$. If the scalar curvature K satisfies $K \geqq 0$ and the distinguisguished normal C^A is parallal in the normal bundle, then M is a cylindrical surface of a $2m$-dimensional Euclidean space E^{2m}.*

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