HONAM MATHEMATICAL JOURNAL
Volume 5, Number 1, September, 1983

On the Extension of a Completely Positive Map on C^*-Algebra

By An-Hyun Kim

I. Introduction.

In [1], Arveson proved that if S is a normed closed selfadjoint subspace of a C^*-algebra A with unit and the unit $I \in A$ and $\mathcal{C}$ is a completely positive map defined on S, then ϕ has a completely positive extension to A. We wish to prove Arveson's theorem in a form which can be applied to algebras of unbounded operators. In this case the unit I is no longer an interior point of the cone of positive elements and the topological structure associated with a C^*-algebra is no longer available.

2. Completely positive Map.

Suppose X is a complex vector space with a conjugate linear involution $x \longrightarrow x^*$ with the properties :

(1) $(\alpha x+y)^* = \bar{\alpha} x^* + y^*$

(2) $x^{**} = x$,

for all $x, y \in X$ and complex numbers α.

We denote by $M(X)$ the space of all finite matrices over X.

Each element $\{x_{ij}\} \in M(X)$ is an array $x_{ij} \in X$ for $i, j=1, 2, \cdots\cdots$ with $x_{ij} \neq 0$ for only finitely many pairs of indices.

We define $*$-operation on $M(X)$ by the relation

$$\{x_{ij}\}^* = \{y_{ij}\} \text{ if } y_{ij} = x^*_{ij}$$

for all $i, j=1, 2 \cdots\cdots$.

we denote by $M(X)_r$ the hermition elements of $M(X)$, those elements such that $\{x_{ij}\} = \{x_{ij}\}^*$.

DEFINITION 2.1. We say a cone Q in $M(X)_r$ is admissible if

(1) Q is a cone in that if $\{x_{ij}\}$, $\{y_{ij}\} \in Q$ and $\lambda, \mu \geq 0$, then $\lambda \{x_{ij}\} + \mu \{y_{ij}\} \in Q$.

(2) If $\{x_{ij}\} \in Q$ and $\{\alpha_{ij} : i, j = 1, 2, \cdots\cdots\}$ is an array of complex numbers such that $\alpha_{ij} \neq 0$ for only finitely many pairs (i, j) of indices and

$$y_{ij} = \sum_{kl=1}^{\infty} \bar{\alpha}_{ki} \alpha_{lj} X_{kl}, \quad \text{then } \{y_{ij}\} \in Q.$$

DEFINITION 2. 2. Suppose Q is an adimssible cone in $M(X)_r$ and K is a complex vector space. A completely positive (with respect to Q) map ϕ of X on K is a linear map ϕ of X into $B(K)$ such that if $\{x_{ij}\} \in Q$ then

$$\sum_{ij=1}^{\infty} \langle f_i \mid \phi(x_{ij}) \mid f_j \rangle \geq 0 \quad \text{for all } f_i \in K, \ i = 1, 2, \cdots\cdots.$$

If Q is an admissible cone in $M(X)_r$, we say an element $x = x^* \in X$ is in $Q (x \in Q)$ if the matrix $\{x \delta_{1i} \delta_{1j}\} \in Q$

DEFINITION 2. 3. A symmetric ($i.e$, $x \in M$ implies $x^* \in M$) subspace M of X is said to be cofinal in X with respect to an admissible cone Q in $M(X)$ if for every $x^* = x \in X$, there is a $y^* = y \in M$ with $y \in Q$ and $y - x \in Q$.

PROPOSITION 2. 4. Suppose M is cofinal in X with respect to an admissible cone Q in $M(X)_r$, then for every $\{x_{ij}\} \in M(X)_r$, there is an element $\{y_{ij}\} \in M(M)_r$ with $\{y_{ij}\} \in Q$ and $\{y_{ij}\} - \{x_{ij}\} \in Q$.

PROPOSITION 2. 5. Suppose X is a real vector space and $\{x_\alpha \in X, a_\alpha \in R : \alpha \in I\}$ is a set of pairs of elements $x_\alpha \in X$ and real numbers a_α indexed by the set I.

Let $S_1 = \{x \in X : x = -\sum_{i=1}^{n} \lambda_i x_{\alpha_i}, \ \lambda_i \geq 0, i = 1, 2, \cdots, n \text{ and } \sum_{i=1}^{n} \lambda_i a_{\alpha_i} < 1, n = 1, \cdots\}$

$V_1 = \{x \in X : x = \sum_{i=1}^{n} \lambda_i x_{\alpha_i}, \ \lambda_i \geq 0, i = 1, 2, \cdots, n \text{ and } \sum_{i=1}^{n} \lambda_i a_{\alpha_i} \leq -1, n = 1, \cdots\}$

If 0 is an internal point of S_1, then one and only one of the folling statements is true :

(a) there exists a real linear functional F on X such that

$F(x_\alpha) + a_\alpha \geq 0$, for all $\alpha \in I$.

(b) $0 \in V_1$.

PROOF. If V_1 is empty, then $F(x) = 0$ satisfies (a), and (b) is false. We will assume that V_1 is not empty. Assume that both (a) and (b) are true. Then from (b) we have $0 = \sum_{i=1}^{n} \lambda_i x_{\alpha_i}$ with $\lambda_i \geq 0$ for $i = 1, 2, \cdots\cdots, n$ and $\sum_{i=1}^{n} \lambda_i a_{\alpha_i} \leq -1$. If F is a real linear functional satisfying (a), then $F(x_{\alpha_i}) + a_{\alpha_i} \geq 0$.
Hence, we have

$$\sum_{i=1}^{n} \lambda_i (F(x_{\alpha_i}) + a_{\alpha_i}) \geq 0, \quad F(\sum_{i=1}^{n} \lambda_i x_{\alpha_i}) \geq -\sum_{i=1}^{n} \lambda_i a_{\alpha_i} \geq 1.$$

Hence, $F(O) \geq 1$ which is a contradiction.

Therefore, statements (a) and (b) can't both be true.

Next suppose 0 is an internal point of S_1, and S_1 and V_1 are disjoint.

Since S_1 and V_1 are convex sets one of which has an internal point, the separation theorem for convex sets assures us that there exists a nonzero linear functional f on X and a real number c such that $f(x) \geq c$ for all $x \in V_1$ and $f(x) < c$ for all $x \in S_1$. Since 0 is an internal point of S_1 and f is nonzero, it follows that $c > 0$.

Let $F(x) = c^{-1} f(x)$ for all $x \in X$. then F satisfies (a). Hence, if (a) is false, S_1 and V_1 are not disjoint and there is an $x \in X$ which lies in both S_1 and V_1, i. e,

$x = -\sum_{i=1}^{n} \lambda_i x_{\alpha_i}$, $x = \sum_{j=1}^{m} \lambda'_j x_{\beta_j}$,

with $\lambda_i, \lambda'_j \geq 0$ for $i = 1, \cdots\cdots, n, j = 1, \cdots\cdots m$ and

$$\sum_{i=1}^{n} \lambda_i a_{\alpha_i} = \delta_1 < 1, \quad \sum_{j=1}^{m} \lambda'_j a_{\beta_j} = \delta_2 \leq -1.$$

We have $\delta = -(\delta_1 + \delta_2) > 0$. then we have

$$0 = \delta^{-1} \sum_{i=1}^{n} \lambda_i x_{\alpha_i} + \delta^{-1} \sum_{j=1}^{m} \lambda_j x_{\beta_j}$$

and $\delta^{-1} \lambda_i \geq 0$, $\delta^{-1} \lambda_j \geq 0$ for $i = 1, \cdots\cdots n$ and $j = 1, \cdots\cdots m$ and Hence, $0 \in V_1$.

Hence, if (a) is false, (b) is true,

and this completes the proof of the proposition. (Q. E. D)

3. Main Theorem.

PROPOSITION 3. 1. Suppose M is a symmetric subspace of X which is cofinal in X with respect to an admissible cone Q in $M(X)$, and ϕ is a completely positive map of M on K.

Suppose $x_0^* = x_0 \in X$ and $x_0 \notin M$. Let M' be the span of M and x_0. Then there exists a completely positive map ϕ' of M' on K which extends ϕ.

PROOF. Suppose the hypotheses of the proposition are satfied.

In order to specify an extension ϕ' of ϕ, it is sufficient to specify $\phi'(x_0)$.

A completely positive extension ϕ' exists if and only if there is a bilinear form $\phi'(x_0)$ such that

(a)$\cdots\cdots \sum_{i,j=1}^{\infty} \langle f_i \mid \phi'(x_0) \mid f_j \rangle + \langle f_i \mid \phi(x_{ij}) \mid f_j \rangle \geq 0$

for all $f_i \in K$, complex number $a_{ij} = \bar{a}_{ji}$ $x_{ij} = x_{ji}^* \in M$ for $i, j = 1, \cdots\cdots$ such that $\{a_{ij} x_0 + x_{ij}\} \in Q$.

Let $\bar{K} \otimes K$ be the linear space of all expression $y = \sum_{i=1}^{n} \bar{f}_i \otimes g_i$

with $f_i, g_i \in K$ for $i = 1, \cdots\cdots n$ and $n = 1, \cdots\cdots$.

On $\bar{K}\otimes K$, we have the relations

$$(\alpha\bar{f}+\bar{f}')\otimes g=\alpha(\bar{f}\otimes g)+\bar{f}'\otimes g.$$
$$\bar{f}\otimes(\alpha g+g')=\alpha(\bar{f}\otimes g)+\bar{f}\otimes g'.$$

We define a $*$-operation on $\bar{K}\otimes K$ by the relation

$$if\ y=\sum_{i=1}^{n}\bar{f}_i\otimes g_i,\quad \text{then } y^*=\sum_{i=1}^{n}\bar{g}_i\otimes f_i.$$

Let Y be the vector space of all $y\in\bar{K}\otimes K$ such that $y=y^*$.

We asserts that each element $y\in Y$ can be expressed in the form

$$y=\sum_{ij=1}^{n}\alpha_{ij}(\bar{f}_i\otimes f_j),\quad \text{where } \alpha_{ij}=\bar{\alpha}_{ji}.$$

Each element $C\in B(K)$ defines a linear functional F_c on $\bar{K}\otimes K$ by the relation $F_c(y)=\sum_{i=1}^{n}\langle f_i\mid C\mid g_i\rangle$, where $y=\sum_{i=1}^{n}\bar{f}_i\otimes g_i$.

If C^* is the hermition adjoint of $C\in B(K)$ $(i.\ e,\ \langle f\mid C^*\mid g\rangle=\langle g\mid C\mid f\rangle$ for all $f,g\in K)$ we have $F_{c^*}(y)=F_c(y^*)$.

It follows that each hermition $C\in B(K)$ defines a real linear functional on Y. In terms of the real vector space Y, the question of whether there is a hermition form$\phi'(x_0)$ satisfying condition (a) is equivalent to the question of whether there exists a real linear functional F on Y such that

(a′) $F(y_a)+a_a\geq 0$, for all $y_a=\sum_{ij=1}^{n}\alpha_{ij}(\bar{f}_i\otimes f_j)\in Y$, $a_a=\sum_{ij=1}^{n}\langle f_i\mid\phi(x_{ij})\mid f_j\rangle$

where $\{\alpha_{ij}x_0+x_{ij}\}\in Q$ and $n=1,2,\cdots\cdots$.

Let S_1 and V_1 be as in proposition 2.5. Then it follows that 0 is an internal point of S_1 and from proposition 2, 5 (a′) has a solution or $0\in V_1$.

Suppose (a′) has no solution and, therefore, $0\in V_1$.

Since $0\in V_1$, there are $y_k=\sum_{ij=1}^{m(k)}\alpha(k)_{ij}\bar{f}_{ik}\otimes f_{jk}$, $k=1,\cdots\cdots n$, $\{x_{ij}^{(k)}\}\in M(M)_n$ with $\{\alpha(k)_{ij}x_0+x_{ij}^{(k)}\}\in Q$ so that $\sum_{k=1}^{n}\lambda_k y_k=0$

$$\text{and } \sum_{k=1}^{n}\sum_{ij=1}^{m(k)}\lambda_k\langle f_{ik}\mid\phi(x_{ij}^{(k)})\mid f_{jk}\rangle\leq -1.$$

Let $\alpha_{(ik)(jl)}=\lambda_k\delta_{kl}\alpha(k)_{ij}$ and $x_{(ik)(jl)}=\lambda_k\delta_{kl}x_{ij}^{(k)}$.

Then combining the pair of indices (ik) into a single index i and, (jl) into j, we have

$$y=\sum_{ij=1}^{m}\alpha_{ij}\ \bar{f}_i\otimes f_j=\sum_{ijkl}\alpha_{(ik)(jl)}\bar{f}_{ik}\otimes f_{jl}=0,$$

$\{\alpha_{ij}x_0+x_{ij}\}\in Q$ with $x_{ij}=x_{ji}^*\in M$ for $i,j=1,\cdots\cdots,m$ and

$$\sum_{ij=1}^{m}\langle f_i\mid\phi(x_{ij})\mid f_j\rangle\leq -1.$$

Since $y=0$, there are $g_i\in K$ for $i=1,\cdots\cdots,r$ (with $r\leq m$) and a matrix $\{r_{ij}: i=1,\cdots\cdots m,\ j=1,\cdots\cdots r\}$ such that

$$f_i=\sum_{j=1}^{r}r_{ij}\ g_j \text{ for } i=1,\cdots\cdots,m \text{ and}$$

$$(r^*ar)_{ij}=\sum_{kl=1}^{m}\bar{r}_{ki}a_{kl}\ r_{lj}=0 \quad \text{for all } i, j=1, \cdots\cdots m.$$

Now from the properties of an admissible cone, we have

$$\{r^*\}\ \{a_{ij}x_0+x_{ij}\}\ \{r\}=\{z_{ij}\}\in Q \qquad \text{with } z_{ij}=\sum_{k,l=1}^{m}\bar{r}_{ki}x_{kl}r_{lj}.$$

Since $\gamma^*a\gamma=0$, the $\{a_{ij}x_0\}$ terms drop out.

Therefore, we have

$$\begin{aligned}\sum_{ij=1}^{r}\langle g_i \mid \phi(z_{ij}) \mid g_j\rangle &=\sum_{ijkl}\langle r_{ki}g_i \mid \phi(x_{kl}) \mid r_{lj}g_j\rangle\\ &=\sum_{kl=1}^{m}\langle \sum_{i=1}^{r}r_{ki}\,g_i \mid \phi(x_{kl}) \mid \sum_{j=1}^{r}r_{lj}g_j\rangle\\ &=\sum_{kl=1}^{m}\langle f_k \mid \phi(x_{kl}) \mid f_l\rangle\\ &=\sum_{ij=1}^{m}\langle f_i \mid \phi(x_{ij}) \mid f_j\rangle \leq -1.\end{aligned}$$

But since $\{z_{ij}\}\in Q$, this contradicts the completely positivity of ϕ on M and we have reached a contradiction.

Hence, inequality (a′) has a solution. (Q. E. D)

THEOREM 3. 2. Suppose M is a symmetric subspace of X which is cofinal in X with respect to an admissible cone Q in $M(X)$, and ϕ is a completely positive map of M on K.

Then ϕ has a completely positive extension to X.

PROOF. The proof follows immediately from an application of Zorn's lemma to the preceding proposition 3, 1.

Let P be the collection of all ordered pairs (M', ϕ'), where M' is a subspace of X which contains M and where ϕ' is a extension of ϕ to M'.

Partially order P by declaring $(M', \phi')\leq(M'', \phi'')$ to mean that $M'\subset M''$ and $\phi''(x)=\phi'(x)$ for all $x\in M'$.

P is not empty since it contains (M, ϕ) and so the Zorn's lemma asserts the existence of a maximal totally ordered subcollection Ω of P.

Let Σ be the collection of all M' such that $(M', \phi')\in\Omega$.

Then Σ is totally ordered by set inclusion, and the union $\bar{M}$ of all members of Σ is a subspace of X.

If $x\in\bar{M}$, then $x\in M'$ for some $M'\in\Sigma$. If $\bar{M}$ were a proper subspace of X, the proposition 3, 1 would give us a further extension, and this would contradict the maximality of Σ. (Q. E. D)

References

[1] William B. Arveson, ***Subalgebra of C^*-algebra,*** Acta. Math. (1969)

[2] N. Dunford and J.T. Schwarz, ***Linear Operators,*** I. Interscience, New York. (1958)

Chonnam National University