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A Note on the Zariski Rings

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§ 0. Introduction

Let A be a commutative ring with identity and let B be a flat A-algebra. If for every A-module M, the mapping $x \rightsquigarrow 1 \otimes x$ of M into M_B is injective, where $M_B = B \otimes_A M$ for an A-module M then B is said to be faithfully flat over A.

A Noetherian topological ring in which the topology is defined by an ideal contained in the Jacobson radical is called a Zariski ring.

In this paper, We shall prove that $\hat{A}$ is faithfully flat over A if and only if A is a Zariski ring for the $\mathfrak{a}$-adic topology where A is a Noetherian ring and $\hat{A}$ the $\mathfrak{a}$-adic completion of A.

§ 1. Faithfully flat modules

Let $f: A \longrightarrow B$ be a ring homomorphism and let $\mathfrak{a}$, $\mathfrak{b}$ be ideals of A, B respectively, then $\mathfrak{a} \subseteqq \mathfrak{a}^{ec}$ and $\mathfrak{b} \supseteqq \mathfrak{b}^{ce}$ where $\mathfrak{a}^e$ is the extension of $\mathfrak{a}$ and $\mathfrak{b}^c$ the contraction of $\mathfrak{b}$.

Furthermore, we know that $\mathfrak{b}^c = \mathfrak{b}^{cec}$ and $\mathfrak{a}^e = \mathfrak{a}^{ece}$.

Lemma 1.1 : Let S be a multiplicatively closed subset of a ring A and let $f: A \to S^{-1}A$ be the natural ring homomorphism, defined by $f(a) = a/1$ for all $a \in A$ Then

(1) Every ideal in $S^{-1}A$ is an extended ideal.

(2) If $\mathfrak{a}$ is an ideal in A, then $\mathfrak{a}^{ec} = \bigcup_{s \in S} (\mathfrak{a} : s)$.

Hence $\mathfrak{a}^e = (1)$ if and only if $\mathfrak{a}$ meets S.

Proof : (1) Let $\mathfrak{b}$ be an ideal in $S^{-1}A$, and let x/s be an element of $\mathfrak{b}$. Then $x/1 \in \mathfrak{b}$, hence $x \in \mathfrak{b}^c$ and therefore $x/s \in \mathfrak{b}^{ce}$. Since $\mathfrak{b} \supseteqq \mathfrak{b}^{ce}$ in any case it follows that $\mathfrak{b} = \mathfrak{b}^{ce}$

(2) $x \in \mathfrak{a}^{ec} = (S^{-1}\mathfrak{a})^c \Longleftrightarrow x/1 = a/s$ for some $a \in \mathfrak{a}$, $s \in S \Longleftrightarrow (x.s - 1.a)t = 0$ for some $t \in S \Longleftrightarrow xst \in \mathfrak{a} \Longleftrightarrow x \in \bigcup_{s \in S}(\mathfrak{a} : s)$.

Lemma 1.2 : Let $A \longrightarrow B$ be a ring homomorphism and let P be a prime ideal of A. Then P is the contraction of a prime ideal of B if and only if $P^{ec} = P$.

Proof : If $P = Q^c$ for a prime ideal Q of B, then $P^{ec} = Q^{cec} = Q^c = P$. Conversely, if $P^{ec} = P$, let S be the image of $A - P$ in B.

Then P^e does not meet S, therefore by Lemma 1.1 its extension in $S^{-1}B$ is a proper ideal and hence is contained in a maximal ideal $\boldsymbol{m}$ of $S^{-1}B$.
If Q is the contraction of $\boldsymbol{m}$ in B, then Q is a prime ideal in B such that $Q \supseteq P^e$ and $Q \cap S = \phi$.
Hence $Q^c = P$ since S is the image of $A - P$.

Definition 1.3 : Let $f : A \rightarrow B$ be a ring homomorphism and let N be a B-module. Then N has an A-module structure defined as follows : if $a \in A$ and $x \in N$ then ax is defined to be $f(a)x$.

This A-module is said to be obtained from N by restriction of scalars.

Lemma 1.4 : Let $f : A \rightarrow B$ be a ring homomorphism and let N be a B-module. Then the homomorphism $g : N \rightarrow N_B$ defined by $g(y) = 1 \otimes y$ is injective and $\mathrm{Im}(g)$ is a direct summand of N_B where $N_B = B \otimes_A N$.

Proof : If $g(y) = 1 \otimes y = 0$ for some $y \in N$, then $y = 0$. Hence g is injective. Now define $p : N_B \rightarrow N$ by $p(b \otimes y) = by$ then p is surjective as a homomorphism. Hence we have a short exact sequence $0 \rightarrow \mathrm{Ker}(p) \rightarrow N_B \xrightarrow{p} N \rightarrow 0$, But $(p \circ g)(y) = p(g(y)) = p(1 \otimes y)$. Therefore we have $p \circ g = 1_N$ Hence the above sequence splits. Thus $N_B = \mathrm{Ker}(p) \oplus \mathrm{Im}(p) = \mathrm{Ker}(p) \oplus \mathrm{Im}(g)$.

Proposition 1.5 : Let $f : A \rightarrow B$ be a ring homomorphism and B a flat A-algebra. Then the following conditions are equivalent :

(i) $\mathfrak{a}^{ec} = \mathfrak{a}$ for all ideals $\mathfrak{a}$ of A.

(ii) ${}^a f : \mathit{Spec}(B) \rightarrow \mathit{Spec}(A)$ is surjective where $\mathit{Spec}(A)$ is the set of all prime ideals of A.

(iii) For every maximal ideal $\boldsymbol{m}$ of A we have $\boldsymbol{m}^e \neq (1)$

(iv) If M is any non-zero A-module then $M_B = B \otimes_A M \neq 0$

(v) For every A-module M, the mapping $x \rightsquigarrow 1 \otimes x$ of M into M_B is injective.

Proof : (i)⇒(ii) For every $P \in Spec(A)$, there exists $Q \in Spec(B)$ such that $P = Q^c$ since $P = Q^c$ if and only if $P^{ec} = Q$ by Lemma. 1.2.

(ii)⇒(iii) is clear since m^e is a prime ideal of B.

(iii)⇒(iv) Let x be a non-zero element of M and let $M' = Ax$. Since B is flat over A it is enough to show that $M'_B \neq 0$.

Define the homomorphism $\phi : A \to Ax$ by $a \rightsquigarrow ax$ for all $a \in A$. then ϕ is surjective and $M' = Ax \cong A/\mathfrak{a}$ for some ideal $\mathfrak{a} \neq (1)$ of A.

Thus $M'_B = B \otimes_A M' \cong B \otimes_A A/\mathfrak{a} \cong B/\mathfrak{a}B \cong B/\mathfrak{a}^e$, but $\mathfrak{a} \subseteq m$ for some maximal ideal m of A and $\mathfrak{a}^e \subseteq m^e \neq (1)$. Hence $M'_B = B/\mathfrak{a}^e \supseteq B/m^e \neq 0$.

(iv)⇒(v) Let M' be the kernel of $M \to M_B$

Since B is flat over A, the sequence

$$0 \to M'_B \to M_B \to (M_B)_B \text{ is exact.}$$

But by Lemma 1.4, the mapping $M_B \to (M_B)_B$ is injective. Hence $M'_B = 0$ and therefore $M' = 0$.

(v)⇒(i) We take $M = A/\mathfrak{a}$ for any ideal $\mathfrak{a}$ of A.

Definition 1.6 : Let B be a flat A-algebra.
If B satisfies one of the conditions of proposition 1.5, then B is said to be faithfully flat over A.

§ 2. Completions

Let G be a topological abelian group and let U be any neighborhood of 0 in G. Then $U+a$ is a neighborhood of a in G, and conversely, every neighborhood of a appears in this form. Thus the topology of G is uniquely determined by the neighborhoods of 0 in G.

Assume that $0 \in G$ has a countable fundamental system of neighborhoods. Then *the completion* $\hat{G}$ of G is defined by the set of all equivalence classes of cauchy sequences.
Hence $\hat{G}$ is an abelian group under addition of classes of cauchy sequences.

For each $x \in G$ the class of the constant sequence (x) is an element $\phi(x)$ of $\hat{G}$ and $\phi : G \to \hat{G}$ is a homomorphism of abelian groups. In general, ϕ is not injective.
We have Ker $\phi = \cap U$ where U runs through all neighborhoods of 0 in G, and so ϕ is injective if and only if G is Hausdorff.

Thus we have a sequence of subgroups

$$G=G_0 \supseteqq G_1 \supseteqq G_2 \supseteqq \cdots\cdots\cdots \supseteqq G_n \supseteqq \cdots\cdots,$$

and $U \subseteqq G$ is a neighborhood of 0 if and only if it contains some G_n, in such topologies the subgroups G_n of G are both open and closed [see[1] and [3]]. Suppose (x_ν) is a Cauchy sequence in G. Then the image ξ_n of x_ν in G/G_n defines a coherent sequence (ξ_n) in the sense that $\theta_{n+1}\ \xi_{n+1}=\xi_n$ for all n, where $\theta_{n+1} : G/G_{n+1} \to G/G_n$ is projection.

Thus $\hat{G}$ can equally well be defined as the set of coherent sequence (ξ_n) with the obvious group structure.

More generally, Consider any sequence of groups $\{A_n\}$ and homomorphisms θ_{n+1} : $A_{n+1} \to A_n$.

We call this an inverse system, and the group of all coherent sequences (a_n), i. e., $a_n \in A_n$ and $\theta_{n+1}\ a_{n+1}=a_n$, is called the inverse limit of the system.

It is usually written $\hat{A} \cong \varprojlim A_n$.

With this notation we have

$$\hat{G} \cong \varprojlim G/G_n.$$

Proposition 2. 1: If $0 \to \{A_n\} \to \{B_n\} \to \{C_n\} \to 0$ is exact sequence of inverse systems then

$$0 \to \varprojlim A_n \to \varprojlim B_n \to \varprojlim C_n \text{ is always exact,}$$

If, moreover, $\{A_n\}$ is surjective system then

$$0 \to \varprojlim A_n \to \varprojlim B_n \to \varprojlim C_n \to 0 \text{ is exact.}$$

Proof: See[[1]].

Corollary 2. 2: Let $0 \to G' \to G \to G'' \xrightarrow{p} 0$ be an exact sequence of groups. Let G have the topology defined by a sequence $\{G_n\}$ of subgroups, and give G', G'' the induced topologies, i. e., by sequences $\{G' \cap G_n\}$, $\{pG_n\}$.

Then $0 \to \hat{G}' \to \hat{G} \to \hat{G}'' \to 0$ is exact.

Proof: Since $\{G'/G' \cap G_n\}$ is surjective system, We can apply proposition 2.1 to the exact sequence of inverse systems

$$0 \to \{G'/G' \cap G_n\} \to \{G/G_n\} \to \{G''/pG_n\} \to 0$$

Hence $0 \to \varprojlim\ G'/G' \cap G_n \to \varprojlim\ G/G_n \to \varprojlim\ G''/pG_n \to 0$ is exact, i. e., $0 \to \hat{G}' \to \hat{G} \to \hat{G}'' \to 0$ is exact.

Corollary 2. 3: $\hat{G}_n$ is a subgroup of $\hat{G}$ and $\hat{G}/\hat{G}_n \cong G/G_n$.

Proof: We can apply Corollary 2.2 with $G' = G_n$, then $G'' = G/G_n$ has the discrete topology so that $\hat{G}'' = G''$.

Corollary 2.4: $\hat{\hat{G}} \cong \hat{G}$.

Proof: Taking inverse limits in the above corollary we deduce $\varprojlim \hat{G}/\hat{G}_n (\cong \hat{\hat{G}}) \cong \varprojlim G/G_n (\cong \hat{G})$.

Definition 2.5: Let G be a group and let $\hat{G}$ be the completion of G. If $\phi : G \to \hat{G}$ is an isomorphiom, then G is said to be complete.

Definition 2.6: Let $\mathfrak{a}$ be an ideal of a ring A and M an A-module. A chain $M = M_0 \supseteqq M_1 \supseteqq M_2 \supseteqq \cdots \supseteqq M_n \supseteqq \cdots$, where the M_n are submodules of M, is called an $\mathfrak{a}$-filtration of M if $\mathfrak{a} M_n \subseteqq M_{n+1}$ for all n, and a stable $\mathfrak{a}$-filtration of M if $\mathfrak{a} M_n = M_{n+1}$ for all sufficiently large n.

Proposition 2.7: (Artin–Rees Lemma)
Let A be a Noetherian ring, $\mathfrak{a}$ an ideal of A, M a finitely-generated A-module and (M_n) a stable $\mathfrak{a}$-filtration of M.

If M' is a submodule of M,
then $(M' \cap M_n)$ is a stable $\mathfrak{a}$-filtration of M'.

Proof: See [[1] and [2]].

§ 3. Main theorems

Let A be a Noetherian ring, $\mathfrak{a}$ an ideal of A, M a finitely-generated A-module and M' a submodule of M.

Then by proposition 2.7 the $\mathfrak{a}$-adic topology of M' coincides with the topology induced by the $\mathfrak{a}$-adic topology of M.

Lemma 3.1: $0 \to M' \to M \to M'' \xrightarrow{p} 0$ be an exact sequence of finitely-generated modules over a Noetherian ring A, Let $\mathfrak{a}$ be an ideal of A, then the sequence of $\mathfrak{a}$-adic completions.

$$0 \to \hat{M}' \to \hat{M} \to \hat{M}'' \to 0 \quad \text{is exact.}$$

Proof: $\{\mathfrak{a}^n M\}$ is the fundamental system of neighborhoods of 0 and $(\mathfrak{a}^n M \cap M')$ is a stable $\mathfrak{a}$-filtration of M' by proposition 2.7. Consider the exact sequence of inverse systems

$$0 \to \{M'/\mathfrak{a}^n M \cap M'\} \to \{M/\mathfrak{a}^n M\} \to \{M''/p\mathfrak{a}^n M\} \to 0$$

then by corollary 2. 2

$$0 \to \hat{M}' \to \hat{M} \to \hat{M}'' \to 0 \quad \text{is exact.}$$

Lemma 3. 2 : For any ring A, if M is a finitely-generated A-module, then $\hat{A}\otimes_A M \to \hat{M}$ is surjective.

If, moreover, A is Noetherian then $\hat{A}\otimes_A M \to \hat{M}$ is an isomorphism.

Proof : If $F \cong A^n$, then we have $\hat{A}\otimes_A F \cong \hat{F}$.

Now assume M is finitely-generated so that we have an exact sequence $0 \to N \to F \to M \to 0$

This gives rise to the commutative diagram

$$\begin{array}{ccccccccc} & & \hat{A}\otimes_A N & \to & \hat{A}\otimes_A F & \to & \hat{A}\otimes_A M & \to & 0 \\ & & \downarrow \gamma & & \downarrow \beta & & \downarrow \alpha & & \\ 0 & \to & \hat{N} & \to & \hat{F} & \underset{\delta}{\to} & \hat{M} & \to & 0 \end{array}$$

in which the top line is exact and δ is surjective by corollary. 2. 2. Since β is an isomorphism this implies that α is surjective.

Assume now that A is Noetherian then N is also finitely-generated so that γ is surjective and, by Lemma 3. 1 the bottom line is exact. Hence α is injective by the four lemma. i. e., α is an isomorphism.

Lemma 3. 3 : If A is Noetherian, $\hat{A}$ its $\mathfrak{a}$-adic completion, Then

(1) $\hat{\mathfrak{a}} = \hat{A}\mathfrak{a} \cong \hat{A}\otimes_A \mathfrak{a}$;

(2) $(\mathfrak{a}^n)\hat{} = (\hat{\mathfrak{a}})^n$;

(3) $\mathfrak{a}^n/\mathfrak{a}^{n+1} \cong \hat{\mathfrak{a}}^n/\hat{\mathfrak{a}}^{n+1}$;

(4) $\hat{\mathfrak{a}}$ is contained in the Jacobson radical of $\hat{A}$.

Proof : (1) Since A is Noetherian, $\mathfrak{a}$ is finitely-generated. Lemma 3. 2 implies that the map $\hat{A}\otimes_A \mathfrak{a} \to \hat{\mathfrak{a}}$ whose image is $\hat{A}\mathfrak{a}$, is an isomorphism.

(2) Now apply (1) to $\mathfrak{a}^n$ and we deduce that $(\mathfrak{a}^n)\hat{} = \hat{A}\mathfrak{a}^n = (\hat{A}\mathfrak{a})^n = (\hat{\mathfrak{a}})^n$.

(3) Applying corollary 2. 3 we now deduce $A/\mathfrak{a}^n \cong \hat{A}/\hat{\mathfrak{a}}^n$ and $A/\mathfrak{a}^{n+1} \cong \hat{A}/\hat{\mathfrak{a}}^{n+1}$, by taking quotients $\mathfrak{a}^n/\mathfrak{a}^{n+1} \cong \hat{\mathfrak{a}}^n/\hat{\mathfrak{a}}^{n+1} = (\hat{\mathfrak{a}})^n/(\hat{\mathfrak{a}})^{n+1}$.

(4) By (2) and corollary 2. 4, we see that $\hat{A}$ is complete for its $\hat{\mathfrak{a}}$-adic topology. Hence for any $x \in \hat{\mathfrak{a}}$

$$(1-x)^{-1} = 1 + x + x^2 + x^3 + \cdots\cdots \quad \text{converges in } \hat{A}.$$

so that $1-x$ is a unit in $\hat{A}$.

This implies that $\hat{\mathfrak{a}}$ is contained in the Jacobson radical of $\hat{A}$

Lemma 3. 4 : Let A be a Noetherian ring, $\mathfrak{a}$ an ideal of A contained in the Jacobson radical and let M be a finitely-generated A-module and $\hat{M}$ the $\mathfrak{a}$-adic completion of M.

Then the $\mathfrak{a}$-adic topology of M is Hausdorff, i.e., $\bigcap_n \mathfrak{a}^n M = 0$.

Proof : Let $\phi : M \to \hat{M}$ be the homomorphism.

Then we put $E = \text{Ker}\ \phi = \bigcap_n \mathfrak{a}^n M$.

The induced topology on E coincides with its $\mathfrak{a}$-adic topology and $\mathfrak{a}E$ is a neighborhood in the $\mathfrak{a}$-adic topology. Therefore $\mathfrak{a}E = E$ and by Nakayama Lemma $E = 0$, i.e., $\bigcap_n \mathfrak{a}^n M = 0$.

Hence the $\mathfrak{a}$-adic topology of M is Hausdorff.

Definition 3.5 : A Zariski ring A is a Noetherian ring equipped with the $\mathfrak{a}$-adic topology such that $\mathfrak{a}$ is contained in the Jacobson radical of A.

Theorem 3.6 : Let A be a Noetherian ring, $\mathfrak{a}$ an ideal of A and $\hat{A}$ the $\mathfrak{a}$-adic completion of A. Then $\hat{A}$ is faithfully flat over A if and only if A is a Zariski ring for the $\mathfrak{a}$-adic topology.

Proof : Assume that $\hat{A}$ is faithfully flat over A.

Then by proposition 1.5-(ii), there exists $P' \in Spec(\hat{A})$ with $P'^c = \boldsymbol{m}$ for any maximal ideal $\boldsymbol{m}$ of A.

If $\boldsymbol{m}'$ is any maximal ideal of $\hat{A}$ containing P' we have $\boldsymbol{m}'^c = \boldsymbol{m}$ as $\boldsymbol{m}$ is a maximal ideal of A.

By Lemma 3.3, $\hat{\mathfrak{a}}$ is contained in the Jacobson radical of $\hat{A}$ and hence $\hat{\mathfrak{a}} \subseteqq \boldsymbol{m}'$.

So we have

$$\mathfrak{a} \subseteqq (\hat{\mathfrak{a}})^c \subseteqq \boldsymbol{m}'^c = \boldsymbol{m}.$$

Hence $\mathfrak{a}$ is contained in the Jacobson radical of A.

Conversely, let M be a finitely-generated A-module and $\hat{M}$ the $\mathfrak{a}$-adic completion of M.

Since $\mathfrak{a}$ is contained in the Jacobson radical of A, by Lemma 3.4, the $\mathfrak{a}$-adic topology of M is Hausdorff,

$$\text{i.e., } \bigcap_n \mathfrak{a}^n M = 0.$$

Therefore the mapping $\phi : M \to \hat{M} \cong \hat{A} \otimes_A M = M_{\hat{A}}$ is injective since $\text{Ker}\ \phi = \bigcap_n \mathfrak{a}^n M = 0$ and by Lemma 3.2. Since we have known that $\hat{A}$ is a flat A-algebra, by proposition 1.5-(v), $\hat{A}$ is faithfully flat over A.

References

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