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Notes on the P. W. Z. integral and the analytic Feynman integral

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1. Introduction.

Let $C_0[a, b]$ will denote Wiener space, that is, the set of R-valued (i. e. real-valued) continuous functions x on $[a, b]$ such that $x(a)=0$. Let $\mathscr{B}$ denote the σ-algebra of Borel sets in $C_0[a, b]$ and let m_1 denote the Wiener measure. One can complete $(C_0[a, b], \mathscr{B}, m_1)$ in the usual way to obtain $(C_0[a, b], \mathscr{W}_1, m_1)$ where $\mathscr{W}_1$ is the class of all Wiener measurable sets.

In a recent paper [9], Johnson and Skoug treat a Banach algebra S of functions on Wiener space which are a kind of stochastic Fourier transform of Borel measure on $L_2[a, b]$ (Precise definition will be given in the next section). For such functions they showed that the analytic Feynman integral, defined by analytic continuation of the Wiener integral, exists, and they give a formula for this Feynman integral.

In [8], Johnson and Skoug treat the scale-invariant measurability in Wiener space. It is known that the P. W. Z. (Paley-Wiener-Zygmund) integral exists for m_1-a. e. $x \in C_0[a, b]$ ([4, 16]). In a Banach algebra S, the existence of the P. W. Z. integral for s-a. e. x is necessary ([9]).

Let $v \in L_2[a, b]$. The purpose of this paper is to formulate the Wiener integral

$$(1.1) \qquad \int_{C_0[a, b]} f\left(\int_a^b v(s)\, \widetilde{dx}(s)\right) dm_1(x)$$

for some Lebesgue measurable functions f on R. By these we can evaluate the Wiener integral easily. At the end of this paper, we give the functions G and K on $C_0[a, b]$ which are Borel measurable and equal m_1-a. e. but such that the analytic Feynman integral of G exists but the analytic Feynman integral of K does not exist.

2. Preliminaries and the P. W. Z. integral.

A subset A of $C_o[a, b]$ is said to be scale-invariant measurable provided ρA is in $\mathcal{B}_1$ for every $\rho>O$. The class $\mathcal{B}$ of scale-invariant measurable sets forms a σ-algebra [8;Proposition 3]. N in $\mathcal{B}$ is said to be scale-invariant null provided $m_1(\rho N) = O$ for every $\rho>0$. A property which holds except on a scale-invariant null set is said to hold scale-invariant almost everywhere (denoted by s-$a.e.$).

Cameron and Martin introduced the concept of scale-invariant null sets in [3], and Johnson and Skoug treated a rather detailed discussion of scale-invariant measurability and its relation with other topics. (see [8])

Let n denote the class of scale-invariant null sets. Then $n \subset \mathcal{B} \subset \mathcal{B}_1$ and $\mathcal{B} \neq \mathcal{B}_1$ [8; Proposition 6]. A function F on $C_o[a, b]$ is said to be scaleinvariant measurable if it is measurable with respect to the σ-algebra $\mathcal{B}$.

Let $a<t_1<\cdots<t_n\leqq b$ and let G be a Lebesgue measurable subset of n-dimensional Euclidean space R^n. It is well known that sets of the form

$$E\equiv\{x \text{ in } C_o[a, b]: (x(t_1), \cdots, x(t_n)) \in G\}$$

are Wiener measurable. In fact E is Wiener measurable if and only if G is Lebesgue measurable [14]. And they are not Borel measurable if G is not Borel measurable. For every $\lambda>0$, $\lambda E=\{x \text{ in } C_o[a, b] : (x(t_1), \cdots, x(t_n)) \in \lambda G\}$ is Wiener measurable since λG is Lebesgue measurable. Thus E is a scale-invarant measurable set.

Let F be a $\mathbb{C}$-valued (i. e. complex-valued) functions on $C_o[a, b]$ which is s-a. e. defined and scale-invariant measurable and such that the Wiener integral

$$J(\lambda)=\int_{C_o[a, b]} F(\lambda^{-\frac{1}{2}}x)\ dm_1(x)$$

exists as a finite number for all $\lambda>O$. If there exists a function $J^*(\lambda)$ analytic in $\mathbb{C}^+=\{\lambda \text{ in } \mathbb{C}: Re\ \lambda>O\}$ such that $J^*(\lambda)=J(\lambda)$ for all $\lambda>O$, then $J^*(\lambda)$ is defined to be the analytic Wiener integral of F over $C_o[a, b]$ with parmeter λ, and, for λ in $\mathbb{C}^+$, we write

$$\int_{C_o[a, b]}^{an\ w_\lambda} F(x)\ dm_1(x)\equiv J^*(\lambda).$$

Let q be real parameter $(q\neq O)$ and let F be a function whose analytic Wiener integral exists for λ in $\mathbb{C}^+$. If the following limit exists, we call it the analytic Feynman integral of F over $C_0[a, b]$ with parameter q, and we write

$$\int_{C_0[a, b]}^{an\ f_q} F(x)\ dm_1(x) = lim \int_{C_0[a, b]}^{an\ w_\lambda} F(x)\ dm_1(x)$$

where λ approaches $-iq$ through $\mathbb{C}^+$.

Let $\{\psi_j\}$ be a complete orthonormal set of R-valued functions of bounded variatton on $[a, b]$. For v in $L_2[a, b]$, let

$$v_n(t) = \sum_{j=1}^{n} \left[\int_a^b v(s)\ \psi_j(s)\ ds \right] \psi_j(t).$$

The P. W. Z. (Paley-Wiener-Zygmund) intergral $\int_a^b v(s)\ \widetilde{dx}(s)$ is defined by

$$\int_a^b v(s)\ \widetilde{dx}(s) \equiv \lim_{n\to\infty} \int_a^b v_n(s)\ dx(s)$$

for all x in $C_0[a, b]$ for which the limit exists.

Now let $m(L_2)$ be the collection of $\mathbb{C}$-valued countably additive measures on $\mathscr{B}(L_2)$ the Borel class of L_2. $m(L_2)$ is a Banach algebra under the total variation norm where convolution is taken as the multiplication in $m(L_2)$.

The Banach algebra S consists of functions H expressible in the form

$$H(x) = \int_{L_2} exp\{ i \int_a^b v(t)\ \widetilde{dx}(t) \}\ d\sigma(v).$$

for s-a. e. x in $C_0[a, b]$ where σ is an element of $m(L_2)$. Cameron and Storvick show that the correspondence $\sigma \to F$ is one-to-one[1;Theorem 2. 1] and carries convolution into pointwise multiplication. Letting $\|H\| \equiv \|\sigma\|$ we have that S is a Banach algebra of functions on Wiener space[1;Section 2]. The analytic Feynman integral exists for every $H \in S$[1;Theorem 5. 1]. Furhter they show that if $\{H_j\}$ is a sequence of elements from S such that $\sum_{j=1}^{\infty} \|H_j\| < \infty$, then $H = \sum_{j=1}^{\infty} H_j$ is in S and the analytic Feynman integral of H is the sum of the analytic Feynman integrals of H_j's[1;Theorem 5. 4].

In a Banach algebra S, the existence of the P. W. Z. integral for s-a. e. x is necessary. So we will prove it.

LEMMA 1. For v in $L_2[a, b]$, the P. W. Z. integral exists for *s-a. e.* $x \in C_0[a,b]$.

Proof. For every $\rho > O$, let $X_k(x) = \int_a^b \phi_k(t)\ d\rho x(t)$. Then $X_k(x) = \rho \int_a^b \phi_k(t)\ dx(t)$. Hence $\{X_k\}$ is a sequence of independent random variables with $N(o, \rho^2)$. Thus $\{(\nu, \phi_k) X_k\}$ is a sequence of independent random variables with $N(O, (v, \phi_k)^2 \rho^2)$. By the Parseval's identity and $v \in L_2[a, b]$, $\sum_{k=1}^{\infty} V((v, \phi_k) x_k) = \sum_{k=1}^{\infty} \rho^2 (v, \phi_k)^2 = \rho^2 \|v\|_2^2 < \infty$.

Thus by the Kolmogorov's Theorem

$$\sum_{k=1}^{\infty} ((v, \phi_k) X_k - E((v, \phi_k) X_k))$$
$$= \lim_{n\to\infty} \sum_{k=1}^{n} (v, \phi_k) \int_a^b \phi_k(s) \ d\rho x(s)$$

exists for m_1-a. e. $x \in C_0[a, b]$. Hence $\int_a^b v(s) \ \widetilde{d\rho x}(s)$ exists for m_1-a. e. $x \in C_0[a, b]$, i. e. $\int_a^b v(s) \ \widetilde{dx}(s)$ exists for s-a. e. $x \in C_0[a, b]$.

Remark. Let $\rho=1$ in the proof of the Lemma 1. Then the P. W. Z. integral exists for m_1*-a. e.* $x \in C_0[a, b]$ for each v in $L_2[a, b]$.

It is easy to show the followings by the similar method to the proof of Lemma 1 and the paper[4; Theorem 2. 8 and 2. 7]. The P. W. Z. integral is essentially independent of the choice of the complete orthonormal set for *s-a. e.* $x \in C_0[a, b]$; further, if v is of bounded variation, the P. W. Z. integral is *s-a. e. equal* to the Riemann-Stieltjes integral $\int_a^b v(s) \ dx(s)$.

3 Main Theorems.

The following Lemma is well known [12]. The result plays a key role here. So we will prove it.

LEMMA 2. If $v \in L_2[a, b]$ and if f is Lebesgue measurable function on R, then

$$\int_{C_0[a,b]} f\Big(\int_a^b v(s) \ \widetilde{dx}(s)\Big) \ dm_1(x) = (2\pi\|v\|_2^2)^{-\frac{1}{2}} \int_{-\infty}^{\infty} f(r) \exp\Big\{-\frac{r^2}{2\|v\|_2^2}\Big\} \ dr.$$

Proof. Let $Y(x) = \int_a^b v(s) \ \widetilde{dx}(s)$. Then Y is normally distributed with $N(O, \|v\|_2^2)$. Thus by the change of the Variables Theorem[7; p. 163]

$$\int_{C_0[a,b]} f\Big(\int_a^b v(s) \ \widetilde{dx}(s)\Big) \ dm_1(x) = \int_{C_0[a,b]} f(Y(x)) \ dm_1(x)$$
$$= (2\pi\|v\|_2^2)^{-\frac{1}{2}} \int_{-\infty}^{\infty} f(r) \ \exp\Big\{-\frac{r^2}{2\|v\|_2^2}\Big\} \ dr.$$

In the following two theorems we formulate the Wiener integral

$$(3.1) \qquad F \equiv \int_{C_0[a,b]} f\Big(\int_a^b v(s) \ \widetilde{dx}(s)\Big) \ dm_1(x)$$

for some Lebesgue measurable functions f on R. It is worth evaluating the Wiener integral (3. 1) for some Lebesgue measurable functions f because the calculation of the Wiener and the analytic Feynman integral is very complicated. Here we shall take $v \in L_2[a, b]$.

THEOREM 3. (a) If $f(r)=1$, then $F=1$.
(b) If $f(r)=|r|$, then $F=(2/\pi)^{\frac{1}{2}}\|v\|_2$.
(c) If $f(r)=r^{2n}$ where n is in N(the set of the positive integers), then

$$F=1\cdot 3\cdot 5\ \cdots(2n-1)\ \|v\|_2^{2n}.$$

(d) If $f(r)=r^{2n-1}$ where n is in N, then $F=0$.
(e) If $f(r)=\sum_{i=0}^{n} a_i\ r^i$, then $F=a_0+\sum_{k=1}^{[\frac{1}{2}n]} a_{2k}\ 1\cdot 3\cdot 5\cdots(2k-1)\ \|v\|_2^{2k}$ where $[x]$ is the greates integer function.

Proof. Statement (a) follows from $\int_{C_0[a,b]}1\ dm_1(x)=m_1(C_0[a,b])=1$. Suppose $f(r)=|r|$, then by Lemma 2

$$F=\frac{2}{(2\pi\|v\|_2^2)^{\frac{1}{2}}}\int_0^\infty r\ exp\{-\frac{r^2}{2\|v\|_2^2}\}\ dr=(2/\pi)^{\frac{1}{2}}\|v\|_2.$$

Thus (b) is proved. Let $f(r)=r^{2n}$ where n is in N. Then

$$F=(2\pi\|v\|_2^2)^{-\frac{1}{2}}\int_{-\infty}^{\infty} r^{2n}\ exp\{-\frac{r^2}{2\|v\|_2^2}\}dr$$
$$=(2\pi)^{-\frac{1}{2}}\|v\|_2^{2n}\int_{-\infty}^{\infty}u^{2n}\ exp\{-\frac{1}{2}u^2\}\ du.$$

Since $(2\pi)^{-\frac{1}{2}}\int_{-\infty}^{\infty}u^{2k}exp\{-\frac{1}{2}u^2\}\ du=1\cdot 3\cdot 5\cdots(2k-1)$, (c) follows. Let $f(r)=r^{2n-1}$ where n is in N. Since $\int_{-\infty}^{\infty}u^{2k-1}\ exp\{-u^2/b\}\ du=0$ where $b>0$ and k is in N, (d) follows. (e) is the immediate consequence of (c) and (d).

REMARK. We do not formulate the formula for $f(r)=r^p$ where p is in R.

THEOREM 4. (a) If $f(r)=exp\{ar\}$ where $a\in\mathbb{C}$, then $F=exp\{\frac{1}{2}a^2\|v\|_2^2\}$.
(a′) If $f(r)=exp\{ir\}$ or $f(r)=exp\{-ir\}$, then $F=exp\{-\frac{1}{2}\|v\|_2^2\}$.
(b) If $f(r)=cos\ ar$ where $a\in\mathbb{C}$, then $F=exp\{-\frac{1}{2}a^2\|v\|_2^2\}$.
(b′) If $f(r)=cos\ ir$, then $F=exp\{-\frac{1}{2}\|v\|_2^2\}$. And if $f(r)=cos\ r$, then $F=exp\{-\frac{1}{2}\|v\|_2^2\}$.
(c) If $f(r)=sin\ ar$ where $a\in\mathbb{C}$, then $F=0$.

Proof. Let $f(r)=exp\{ar\}$ where $a\in\mathbb{C}$. Then

$$F=(2\pi\|v\|_2^2)^{-\frac{1}{2}}\int_{-\infty}^{\infty}exp\{ar-\frac{r^2}{2\|v\|_2^2}\}dr$$

by Lemma 2. Since $\int_{-\infty}^{\infty}exp\{-(\alpha u^2+\beta u)\ du=(\pi/\alpha)^{\frac{1}{2}}\ exp\{\beta^2/4\alpha\}$ where $\alpha>0$ and β is real or imaginary[17; p. 161], $F=exp\{\frac{1}{2}a^2\ \|v\|_2^2\}$. Thus (a) is proved. Statement (a′) follows from (a) with $a=\pm i$. Suppose $f(r)=cos\ ar$ where $a\in\mathbb{C}$. Since $cos\ ar=\frac{1}{2}\{exp\ \{iar\}+exp\{-iar\}\}$, $F=exp\{-\frac{1}{2}a^2\|v\|_2^2\}$

by Lemma 2 and the linearity and (a). This proves (b). From the statement (b) with $a=i$ and $a=1$ respectively, the statement (b′) follows. Since $\sin ar = \{\exp\{iar\} - \exp\{-iar\}\}/(2i)$ where $a \in \mathbb{C}$, the statement (c) follows by Lemma 2 and the linearity and (a).

REMARK. We can make the formula F for other Lebesgue measurable functions f (for example, the hyperbolic functions) similarly.

We shall say that two functionals $F(x)$ and $G(x)$ are *equal s-a. e.* if $F(x)=G(x)$ except for a scale-invariant null sets, in other words, if for each $\rho>0$ the equation $F(\rho x)=G(\rho x)$ holds for m_1-a. e. $x \in C_0[a, b]$. We denote this equivalence relation between functionals by $F \approx G$[1; p. 21]. Here we can easily see that $\approx$ is an equivalence relation. It is the appropriate equivalence relation for the analytic Feynman integral.

Johnson and Skoug give an example such that $G=K$ m_1-*a. e.* but the analytic Wiener and Feynman integral are not equal. Here we give another example such that G and K are Borel measurable functions on $C_0[a, b]$ and *equal* m_1-*a. e.* but such that the analytic Feynman integral of G exists but the analytic Feynman integral of K does not exist. We finish this paper by showing that it is.

For each $\alpha>0$, let C_λ be a subset of $C_0[a, b]$ such that $m_1(C_1)=1$, $C_\alpha \cap C_\beta = \phi$ for $\alpha \neq \beta$ and $\lambda C_\beta = C_{\lambda\beta}$[8]. Let $G \equiv 1$ and $K(x) \equiv \chi_{C_1}(x)$ where $x \in C_0[a, b]$. Then $G(x)=K(x)$ for m_1-*a. e.* $x \in C_0[a, b]$ since $m_1(C_1)=1$. And G and K on $C_0[a, b]$ are Borel measurable since C_1 is Borel measurable. Here

$$\int_{C_0[a, b]}^{an\ w_\lambda} G(x)\ dm_1(x) = 1 = \int_{C_0[a, b]}^{an\ f_q} G(x)\ dm_1(x)$$

for all $\lambda \in \mathbb{C}^+$ and all real $q \neq 0$. But

$$K(\lambda^{-\frac{1}{2}}x) = \begin{cases} 0 & (\lambda \neq 1,\ \lambda>0) \\ 1 & (\lambda=1) \end{cases}$$

for m_1-*a. e.* $x \in C_0[a, b]$. Hence there doesn't exist $J^*(\lambda)$ analytic in $\mathbb{C}^+$ such that

$$J^*(\lambda) = \int_{C_0[a, b]} K(\lambda^{\frac{1}{2}}x)\ dm_1(x)$$

for all $\lambda>o$, since $J^*(1)=1$ and $J^*(\lambda)=0$ for $\lambda \neq 1$ and $\lambda>0$. Thus the analytic - Wiener and the analytic Feynman integral of K does not exist.

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