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Common Fixed Point Theorem for Densifying Mappings in Probabilistic Metric Spaces

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Generalizing the Kuratowski's measure of noncompactness, Bocşan and Constantin [3] introduced the notion of the Kuratowski's function on a probabilistic metric space (PM-space). The concept of probabilistic densifying (also called condensing) mapping was introduced in [2] by Bocşan and some fixed point theorems for these mappings have been proved in [2], [5] and recently in [7].

In the present paper we introduce the concept of joint sequence of iterates at a point and prove a common fixed point theorem for a pair of commuting densifying mappings in PM-spaces. This extends the result of Ranganathan and Gupta [6] to PM-spaces. The result in [7] is obtained as a special case of our result.

Throughout this note (X, F) stands for a PM-space and (X, F, t) for a complete Menger space. It is known that every metric space (S, d) is a PM-space with $F_{p,q}(x) = H(x-d(p, q))$ where

$$H(x) = \begin{cases} 0, & x \leqq 0 \\ 1, & x > 0. \end{cases}$$

Let A be a nonempty subset of X.

Definition 1 ([3]). The function $D_A(\cdot)$ defined by

$$D_A(u) = \sup_{x<u} \inf_{p,q \in A} F_{p,q}(x)$$

is called the probabilistic diameter of A (see also [4]). A is called probabilistic bounded if $\text{Sup } u \in RD_A(u) = 1$. The mapping $\alpha_A(\cdot)$ defined by

$\alpha_A(u) = \sup\{ \varepsilon \leqq O: \exists$ a finite cover A of A such that $D_A(u) \geqq \varepsilon$ for all $A \varepsilon A\}$ is called the Kuratowski's function for the probabilistic bounded subset A of X.

The following properties of α_A are proved in [3]

1. $\alpha_A \varepsilon \mathcal{L}$, the set of all distribution functions F;

2. $\alpha_A(u) \geqq D_A(u)$;

3. if $A \subset B \subset X$ then $\alpha_A(u) \geqq \alpha_B(u)$;

4. $\alpha_{A \cup B}(u) = \min\{\alpha_A(u), \alpha_B(u)\}$;

5. let $\bar{A}$ be the closure of A in the (ε, λ)- topology on X, then $\alpha_A(u) = \alpha_{\bar{A}}(u)$.

Definition 2([2]). Let (X, F) be a probabilistic metric space. A mapping T from a subset M of X to M is called probabilistic densifying if $\alpha_A < H$, where $A \subset M$, implies $\alpha_{T(A)} > \alpha_A$.

Definition 3([1]). A mapping $T : X \to X$ is called a probabilistic weakly ϕ-contractive iff the following two conditions are satisfied:

1. for every $A \subset X$, $\alpha_A < H$ implies $\alpha_{T(A)} > \alpha_A$;
2. for every $p, q \in X$, $p \neq q$

$$\phi(T(p), T(q)) > \phi(p, q)$$

where ϕ is a τ-continuous mapping [1] of $X \times X$ into L. It is to be noted that τ-continuity coincides with upper semicontinuity [5, page 266].

The following lemma is obvious:

Lemma: Let (X, F, t) be a complete Menger space and T_1, T_2 be two mappings from X to itself. T_1 and T_2 are probabilistic densifying mapping iff for every pair of probabilistic bounded subsets A and B of X

$$\alpha_{(T_1(A) \cup T_2(B))} > \alpha_{A \cup B} \text{ whenever } \alpha_{A \cup B} < H.$$

We now introduce the following

Definition 4. Let $S = \{T_1, T_2\}$ be a pair of mappings from a Menger space (X, F, t) to itself. For $u_o \in X$, let $u_n = T_1 u_{n-1}$ if n is odd and, $u_n = T_2 u_{n-1}$ if n is even, then the sequence

$$J_{S(u_o)} = \{u_o, T_1 u_o, T_2 T_1 u_o, \cdots\}$$

is called the joint sequence of iterates of S at u_o.

Theorem. Let $S = \{T_1, T_2\}$ be a pair of commuting, probabilistic densifying mappings from a complete Menger space (X, F, t) to itself such that $\sup_{u<1} t(u, u) = 1$. Further, let $T_1 T_2$ be a weakly ϕ-contractive and for some $u_o \in X$ the joint sequence of iterates $J_{S(u_o)}$ of S at u_o be bounded. Then T_1 and T_2 have a unique common fixed point in X.

Proof. Let $A=\{u_0, T_1u_0, T_2T_1u_0, \cdots\}$, $B=\{u_0, T_2T_1u_0, T_2T_1T_2T_1u_0, \cdots\}$ and $C=\{T_1u_0, T_1T_2T_1u_0, \cdots\}$ so that $A=B\cup C$ and $A=T_1(B)\cup T_2(C)\cup\{u_0\}$. Therefore, if $\alpha_A<H$

$$\begin{aligned}\alpha_A &= \alpha_{T_1(B)\cup T_2(C)\cup\{u_0\}}\\ &= \min\{\alpha_{T_1(B)\cup T_2(C)}, \alpha_{\{u_0\}}\}\\ &= \alpha_{T_1(B)\cup T_2(C)}\\ &> \alpha_{B\cup C} \text{ (in view of the lemma)}\\ &= \alpha_A, \text{ a contradiction. Hence } A \text{ is compact.}\end{aligned}$$

We now define a mapping $f: \bar{A}\to R$ by $f(u)=\phi(u, T_1T_2u)$. Then f is upper semi-continuous on A as ϕ is $u.s.c.$ So f has a maximal value at some $z\in\bar{A}$.

Now $T_1T_2(\bar{A})=T_1(T_2(A))\subset T_1(T_2(A))\subset T_1(T_2(A))\subset\bar{A}$. Hence $T_1T_2(z)\in A$. Also, for $z\neq T_1T_2(z)$

$$\begin{aligned}f(T_1T_2(z)) &= \phi(T_1T_2(z), T_1T_2T_1T_2(z))\\ &> \phi(z, T_1T_2(z))\\ &= f(z), \text{ a contradiction to the maximality of } f \text{ at } z.\end{aligned}$$

Hence $T_1T_2(z)=z$, that is, z is the fixed point of T_1T_2. Uniqueness of z can be seen easily.

Further $z=T_1T_2(z)\Rightarrow T_1(z)=T_1T_1T_2(z)$

$$= T_1T_2(T_1z).$$

Thus $T_1(z)$ is a fixed point of T_1T_2.

But z also has been shown to be a fixed point of T_1T_2. Therefore

$z=T_1(z)$. Similarly $z=T_2(z)$.

Hence z is the unique common fixed point of T_1 and T_2.

Remark. By taking $T_1=T_2$ in the above theorem we get the result in [7] as a special case.

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