HONAM MATHEMATICAL JOURNAL
Volume 5, Number 1, September, 1983

Fixed Point Theorems for Commuting Mappings in Probabilistic Metric Spaces

By S. L. Singh & B. D. Pant

§ 1. Sehgal [20] initiated the study of fixed points of contraction mappings on probabilistic metric spaces (PM-spaces) (cf. also [21] and [22]). Ćirić [2] introduced the notion of 'generalized contraction' on a PM-space (see Remark 1 below). For fixed point theorems in PM-spaces refer to [5], [6], [7], [28] and references thereof. On the other hand Jungck [9] generalized the Banach contraction principle by introducing a contraction condition for a pair of commuting self-mappings on a metric space. He also pointed out the potential of commuting mappings for generalizing fixed point theorems in metric spaces (cf. also [10]). The essential part of Jungck's result [9] may be stated as follows:

A pair of commuting self-mappings P and T on a complete metric space (M, d) possesses a unique common fixed point, if T is continuous, $P(M) \subseteq T(M)$ and there exists a constant $h \in (0, 1)$ such that $d(px, py) \leq hd(Tx, Ty)$ for every $x, y \in M$.

This result has been extensively generalized by demanding the pair to satisfy general type of contractive or functional conditions (*e.g.* see [3], [11], [13], [14], [15], [16], [17], [18], [23], [24], [25], [26] and [32]). Further generalizations have also been suggested by considering contractive type and functional conditions for three self-mappings of a metric space-one of the mappings commuting with the other two- (see [4], [12], [27], [29], [30] and [31]).

In this paper we combine the ideas of Ćirić [2] and Singh and Singh [29], and introduce the notion of 'generalized contraction triplet' for three self- mappings on a PM-space (cf. Definition 1). In § 2 of this paper, three fixed point theorems are proved for such mappings. One of the results is a fixed point theorem for mappings from a product space $X \times X$ to a PM-space X. Finally, in § 3, we present convergence theorems for three sequences of mappings and their common fixed points.

A PM-space is an ordered pair $(X, \mathcal{F})$ consisting of a non-empty set X and a mapping $\mathcal{F}$ from X x X to $\mathcal{L}$, the collection of all distribution functions. The value of $\mathcal{F}$ at $(p, q) \in X \times X$ is represented by $F_{p,q}$. The functions $F_{p,q}$ are assumed to satisfy the following conditions :

(i) $F_{u,v}(x)=1$ for all $x>0$, iff $u=v$:

(ii) $(F_{u,v}(o)=0$:

(iii) $F_{u,v}=F_{v,u}$:

(iv) if $F_{u,v}(x)=1$ and $F_{v,w}(y)=1$ then $F_{u,w}(x+y)=1$.

A Menger space is a triplet $(X, \mathcal{F}, t)$, where $(X, \mathcal{F})$ is a PM-space and t-norm (or T-norm [19] or Δ-norm [21]) t is such that (iv) is replaced by

(iv′) $F_{u,w}(x+y) \geq t\{F_{u,v}(x),\ F_{v,w}(y)\}$

for all $x \geq 0$, $y \geq 0$. For topological preliminaries refer to Schweizer and Sklar [19], (see also [6]).

Throughout this paper $(X, \mathcal{F})$ stands for a PM-space and (M, d) for a metric space.

DEFINITION 1. Three mappings P. Q. T on a PM-space $(X, \mathcal{F})$ will be called a 'generalized contraction triplet' $(P. Q : T)$ iff there exists a constant $h \in (0, 1)$ such that for every u, v in X,

(1) $F_{Pu,Qv}(hx) \geq \min\{F_{Tu,Tv}(x),\ F_{Pu,Tu}(x),\ F_{Qv,Tv}(x),\ F_{Pu,Tv}(2x),\ F_{Qv,Tu}(2x)\}$

holds for all $x>0$.

DEFINITION 2. Let P, Q and T be mappings from X (resp. M) to itself. If there exists a point u_0 in X (resp. M) and a sequence $\{u_n\}$ in X (resp. M) such that

(2) $Tu_{2n+1}=Pu_{2n},\ \ Tu_{2n+2}=Qu_{2n+1}$ for $n=0, 1, 2, \cdots$,

then the space X (resp. M) will be called $(P. Q : T)$-orbitally complete with respect to u_0 or simply $(P. Q : T(u_0))$-orbitally complete if the closure of $\{Tu_n : n=1, 2, \cdots\}$ is complete.

If $P=Q$ and T is an identity mapping on X (resp. M) then the space X(resp. M) will be called $P(u_0)$-orbitally complete X (resp. M) is called P-orbitally complete if it is $P(u_0)$-orbitally complete for every u_0 in X (resp. M), [1, 8].

DEFINITION 3. T will be called $(P, Q : T(u_0))$-orbitally continuous if the restriction of T on the closure of $\{Tu_n : n=1, 2, \cdots\}$ is continuous.

§ 2. We shall need the following lemma.

LEMMA [28]. Let $\{y_n\}$ be a sequence in a Menger space $(X, \mathcal{F}, t)$, where t is continuous and satisfies $t(x, x) \geq x$ for every $x \in [0, 1]$. If there exists on $h \in (0, 1)$ such that

$$F_{y_n, y_{n+1}}(hx) \geq F_{y_{n-1}, y_n}(x), \quad n=1, 2, \cdots,$$

for all $x \geq 0$, then $\{y_n\}$ is a Cauchy sequence in X.

THEOREM 1. Let $(X, \mathcal{F}, t)$ be a Menger space, where t is continuous and satisfies $t(x, x) \geq x$ for every $x \in [0, 1]$, and $P, Q, T: X \to X$. Further, let $(P, Q: T)$ be a generalized contraction triplet and T commute with each of P and Q. If there exists a point u_0 in X such that X is $(P, Q: T(u_0))$-orbitally complete and T is $(P, Q: T(u_0))$-orbitally continuous, then P, Q and T have a unique common fixed point and $\{Tu_n\}$ converges to the fixed point.

PROOF. By (1)

$$F_{Tu_{2n+1}, Tu_{2n+2}}(hx) = F_{Pu_{2n}, Qu_{2n+1}}(hx) \geq \min\{F_{Tu_{2n}, Tu_{2n+1}}(x), F_{Tu_{2n+1}, Tu_{2n}}(x), F_{Tu_{2n+2}, Tu_{2n+1}}(x), F_{Tu_{2n+1}, Tu_{2n+1}}(2x), F_{Tu_{2n+2}, Tu_{2n}}(2x)\},$$

giving

$$F_{Tu_{2n+1}, Tu_{2n+2}}(hx) \geq F_{Tu_{2n}, Tu_{2n+1}}(x),$$

since

$$F_{Tu_{2n+2}, Tu_{2n}}(2x) \geq \min\{F_{Tu_{2n+2}, Tu_{2n+1}}(x), F_{Tu_{2n+1}, Tu_{2n}}(x)\},$$

Similarly

$$F_{Tu_{2n+2}, Tu_{2n+3}}(hx) \geq F_{Tu_{2n+1}, Tu_{2n+2}}(x).$$

In general,

$$F_{Tu_{n+1}, Tu_{n+2}}(hx) \geq F_{Tu_n, Tu_{n+1}}(x).$$

so by the above lemma $\{Tu_n\}$ is a Cauchy sequence, and converges to a point z in X. We now prove that

$$Qz = Tz.$$

Let $U_{Qz}(\varepsilon, \lambda)$ be a neighborhood of Qz. Since, by the continuity condition on T, $TTu_{2n} \to Tz$ and $TTu_{2n+1} \to Tz$, there exists on integer $K = K(\varepsilon, \lambda)$ such that

(3) $n \geq K$ implies $F_{TTu_{2n}, Tz}\left(\frac{1-h}{2h}\varepsilon\right) > 1-\lambda$ and $F_{TTu_{2n+1}, Tz}\left(\frac{1-h}{2h}\varepsilon\right) > 1-\lambda$.

By (1),

$$F_{TTu_{2n+1}, Qz}(\varepsilon) = F_{PTu_{2n}, Qz}(\varepsilon) \geq \min\{F_{TTu_{2n}, Tz}(\varepsilon/h), F_{TTu_{2n+1}, TTu_{2n}}(\varepsilon/h),$$

$$F_{Qz,\,Tz}(\varepsilon/h),\ F_{TTu_{2n+1},\,Tz}(2\varepsilon/h),\ F_{Qz,\,TTu_{2n}}(2\varepsilon/h)\} \geq \min$$
$$\{F_{TTu_{2n},\,Tz}(\varepsilon/h),\ F_{TTu_{2n+1},\,Tz}(\varepsilon/2h),\ F_{TTu_{2n},\,Tz}(\varepsilon/2h),$$
$$F_{Qz,\,TTu_{2n+1}}(\frac{1+h}{2h}\varepsilon),\ F_{TTu_{2n+1},\,Tz}(\frac{1-h}{2h}\varepsilon),$$
$$F_{TTu_{2n+1},\,Tz}(2\varepsilon/h),\ F_{Qz,\,TTu_{2n+1}}(\varepsilon/h),\ F_{TTu_{2n+1},\,TTu_{2n}}(\varepsilon/h)\},$$

giving

$$F_{TTu_{2n+1},\,Qz}(\varepsilon) \geq \min\{F_{TTu_{2n},\,Tz}(\varepsilon/2h),\ F_{TTu_{2n+1},\,Tz}(\frac{1-h}{2h}\varepsilon),$$
$$F_{TTu_{2n+1},\,TTu_{2n}}(\varepsilon/h)\}$$
$$\geq \min\{F_{TTu_{2n},\,Tz}(\frac{\varepsilon}{2h}),\ F_{TTu_{2n+1},\,Tz}(\frac{1-h}{2h}\varepsilon),$$
$$F_{TTu_{2n+1},\,Tz}(\frac{1-h}{2h}\varepsilon),\ F_{Tz,\,TTu_{2n}}(\frac{1-h}{2h}\varepsilon)\}$$
$$= \min\{F_{TTu_{2n+1},\,Tz}(\frac{1-h}{2h}\varepsilon),\ F_{Tz,\,TTu_{2n}}(\frac{1-h}{2h}\varepsilon)\}$$

So by (3),

$$F_{TTu_{2n+1},\,Qz}(\varepsilon) > 1-\lambda \quad \text{for all } n \geq K.$$

Consequently $Tz = Qz$. Similarly $Tz = Pz$.

To prove $Tz = z$, let $U_{Tz}(\varepsilon, \lambda)$ be a neighbourhood of Tz. Since $\{Tu_n\}$ is a Cauchy sequence, there exists an integer $K = K(\varepsilon, \lambda)$ such that

$$F_{Tu_{2n},\,Tu_{2n+1}}(\frac{1-h}{2h}\varepsilon) > 1-\lambda \quad \text{for all } n \geq K.$$

By (1),

$$F_{Tu_{2n+1},\,Tz}(\varepsilon) = F_{Pu_{2n},\,Qz}(\varepsilon)$$
$$\geq \min\{F_{Tu_{2n},\,Tz}(\varepsilon/h),\ F_{Tu_{2n+1},\,Tu_{2n}}(\varepsilon/h),\ 1,$$
$$F_{Tu_{2n+1},\,Tz}(2\varepsilon/h),\ F_{Tz,\,Tu_{2n}}(2\varepsilon/h)\}$$
$$\geq \min\{F_{Tu_{2n},\,Tu_{2n+1}}(\frac{1-h}{2h}\varepsilon),\ F_{Tu_{2n+1},\,Tz}(\frac{1+h}{2h}\varepsilon),$$
$$F_{Tu_{2n+1},\,Tu_{2n}}(\varepsilon/h),\ F_{Tu_{2n+1},\,Tz}(2\varepsilon/h),$$
$$F_{Tz,\,Tu_{2n+1}}(\varepsilon/h),\ F_{Tu_{2n+1},\,Tu_{2n}}(\varepsilon/h)\},$$

giving

$$F_{Tu_{2n+1},\,Tz}(\varepsilon) \geq F_{Tu_{2n},\,Tu_{2n+1}}(\frac{1-h}{2h}\varepsilon).$$

so

$$F_{Tu_{2n+1},\,Tz}(\varepsilon) > 1-\lambda \quad \text{for all } n \geq K.$$

so,

$$z = Tz, \quad \text{since } Tu_{2n+1} \rightarrow z.$$

To prove the uniqueness of z as a common fixed point of P, Q and T, let y be another common fixed point.

For some $x > 0$, we have by (1),

$$F_{y,\,z}(x) > F_{y,\,z}(x/h).$$

Thus

$$F_{y,z}(x) \geq F_{y,z}(x/h^n) \to 1 \text{ as } n \to \infty,$$

proving $y = z$.

REMARK 1. If $P = Q$ and T is an identity mapping in (1), then generalized contraction (on a PM-space) introduced by Ćirić' [2] is obtained. Hence, in this case Ćirić's result [Th. 1, 2] is obtained as a corollary to the above theorem.

COROLLARY. Let (M, d) be a metric space, and P, Q and T be mappings from M to M such that $PT = TP$ and $TQ = QT$. further, let M be $(P, Q : T(u_0))$ -orbitally complete and T be $(P, Q : T(u_0)$-orbitally continuous. If there exists a constant $h \in (0, 1)$ such that

$$d(pu, Qv) \leq h \max \{d(Tu, Tv), , d(Pu, Tu), d(Qv, Tv), \\ \tfrac{1}{2} d(Pu, Tv), \tfrac{1}{2} d(Qv, Tu)\}$$

for all $u, v, \epsilon M$, then P, Q and T have a unique common fixed point and $\{Tu_n\}$ converges to the fixed point.

PROOF. It may be completed following Ćirić [2, Cor. 1. 1].

THEOREM 2. Let $(X, \mathcal{F}, t)$ be a complete Menger space, where t is continuous and satisfies $t(x, x) \geq x$ for every $x \in [0, 1]$, and $P, Q, T: X \to X$, Further, let $(P, Q : T)$ be a generalized contraction triplet, $PT = TP$, $QT = TQ$ and $P(X) \cup Q(X) \subseteq T(X)$. If T is continuous, then P, Q and T have a unique common fixed point.

PROOF. We take u_0 in x and construct a sequence $\{u_n\}$ in x in the following way :

$$Tu_{2n+1} = Pu_{2n}, \quad Tu_{2n+2} = Qu_{2n+1}, \quad n = 0, 1, 2 \cdots.$$

This can be done since $P(X) \cup Q(X) \subseteq T(X)$. Now the proof of Theorem 1 works.

Now we apply Theorem 2 to establish the following result.

THEOREM 3. Let $(X, \mathcal{F}, t)$ be a complete Menger space, where t is continuous and satisfies $t\ (x, x) \geq x$ for every $x \in [0, 1]$, and P, Q, T three mappings from the product space $X \otimes X$ to X such that

$$P(X \otimes \{v\}) \cup Q(X \otimes \{v\}) \subseteq T(X \otimes \{v\}), \quad P(T(u, v), v) = T(P(u, v), v)$$

and

$$Q(T(u, v), v) = T(Q(u, v), v)$$

for all u, v in X. Suppose that

$$F_{P(u,v),\,Q(u',v')}(hx)$$

(3.1) $$\geq \min\{F_{T(u,v),\,T(u',v')}(x),\ F_{P(u,v),\,T(u,v)}(x),\ F_{Q(u',v'),\,T(u',v')}(x),\ F_{P(u,v),\,T(u',v')}(2x),\ F_{Q(u',v'),\,T(u,v)}(2x),\ F_{v,v'}(x)\}$$

for all u, v, u', v' in X, for all $x>0$ and for some constant $h \in (0,1)$. If T is continuous, then there exists exactly one point b in X such that

$$P(b,b)=Q(b,b)=T(b,b)=b.$$

PROOF. Let $v=v'$ in (3.1). Then

$$F_{P(u,v),\,Q(u',v)}(hx) \geq \min\{F_{T(u,v),\,T(u',v)}(x),\ F_{P(u,v),\,T(u,v)}(x),\ F_{Q(u',v),\,T(u',v)}(x),\ F_{P(u,v),\,T(u',v)}(2x),\ F_{Q(u',v),\,T(u,v)}(2x)\}$$

Therefore for a fixed v in X, Theorem 2 yields that there exists a unique $u(v)$ in X such that

(3.2) $$P(u(v),v)=Q(u(v),v)=T(u(v),v)=u(v).$$

Therefore for any v, v' in X we have by (3.1),

$$F_{u(v),\,u(v')}(hx)=F_{P(u(v),v),\,Q(u(v'),v')}(hx)$$
$$\geq \min\{F_{u(v),\,u(v')}(x),\ F_{u(v),\,u(v)}(x),\ F_{u(v'),\,u(v')}(x),\ F_{u(v),\,u(v')}(2x),\ F_{u(v'),\,u(v)}(2x),\ F_{v,v'}(x)\}.$$

giving

$$F_{u(v),\,u(v')}(hx) \geq F_{v,v'}(x).$$

Since this is true for all $x>0$, $u(\cdot)$ is a contraction on the complete Menger space X. so there exists a unique b in X such that $u(b)=b$. Hence, by (3.2),

$$P(b,b)=Q(b,b)=T(b,b)=b.$$

§ **3.** Let P and P_n $(n=1,2,\cdots)$ be mappings on a PM-space. If $P_n \to P$ uniformly on X then Ćirić [2]. Istrăţescu [6] and others (see also references of [2] and [6]) have investigated the conditions under which the sequence of fixed points of P_n $(n=1,2,\cdots)$ converges to the fixed point of P. Similar investigations have been made by Istrăţescu and Săcuiu [7] in the case of two sequences of mappings on a PM-space. Istrăţescu [6, page 342] has also proved a convergence theorem, if $\{P_n\}$ is pointwise convergent to P. This section offers (uniform and pointwise) convergence theorems for three sequences of mappings.

THEOREM 4. Let $(X, \mathcal{F}, t)$ be a Menger space, where t is continuous and satisfies $t(x, x) \geq x$ for every $x \in [0, 1]$. Let P_n, Q_n and T_n be self-mappings of X with a common fixed point Z_n, $n=1, 2, \cdots$. Let $(P, Q : T)$ be a generalized contraction triplet on X with z as their common fixed point. If the sequences $\{P_n\}$, $\{Q_n\}$ and $\{T_n\}$ converge uniformly to P, Q and T respectively on $\{z_n : n=1, 2, \cdots\}$, then $z_n \to z$.

PROOF. For any n,

$$F_{z_n, z}(\varepsilon) = F_{P_n z_n, Qz}(\frac{1-h}{2}\varepsilon + \frac{1+h}{2}\varepsilon)$$
$$\geq \min\{F_{P_n z_n, Pz_n}(\frac{1-h}{2}\varepsilon),\ F_{Pz_n, Qz}(\frac{1+h}{2}\varepsilon)\}$$
$$\geq \min\{F_{P_n z_n, Pz_n}(\frac{1-h}{4}\varepsilon),\ F_{Pz_n, Qz}(\frac{1+h}{2}\varepsilon)\}. \quad (4.1)$$

By (1),

$$F_{Pz_n, Qz}(\frac{1+h}{2}\varepsilon) = F_{Pz_n, Qz}(\frac{1+h}{2h}\varepsilon),\ F$$
$$\geq \min\{F_{Tz_n, Tz}(\frac{1+h}{2h}\varepsilon),\ F_{Pz_n, Tz_n}(\frac{1+h}{2h}\varepsilon),$$
$$F_{Qz, Tz}(\frac{1+h}{2h}\varepsilon),\ F_{Pz_n, Tz}(\frac{1+h}{h}\varepsilon),$$
$$F_{Qz, Tz_n}(\frac{1+h}{h}\varepsilon)\},$$

giving

$$F_{Pz_n, Qz}(\frac{1+h}{2}\varepsilon) \geq \min\ \{F_{Tz_n, Tz}(\frac{1+h}{2h}\varepsilon),\ F_{Pz_n, Tz_n}(\frac{1+h}{2h}\varepsilon)\},$$

since $Qz = Tz = z$

$$\geq \min\{F_{Tz_n, T_n z_n}(\frac{1-h}{2h}\varepsilon),\ F_{T_n z_n, Tz}(\frac{1+2h}{2h}\varepsilon),$$
$$F_{Pz_n, P_n z_n}(\frac{1+h}{4h}\varepsilon),\ F_{P_n z_n, Tz_n}(\frac{1+h}{4h}\varepsilon)\}$$
$$\geq \min\{F_{Tz_n, T_n z_n}(\frac{1-h}{4}\varepsilon),\ F_{z_n, z}(\frac{1+2h}{2h}\varepsilon), \quad (4.2)$$
$$F_{Pz_n, P_n z_n}(\frac{1-h}{4}\varepsilon)\},$$

since $P_n z_n = T_n z_n = z_n$.

Since $\{P_n\}$ and $\{T_n\}$ converge uniformly to P and T, for $\varepsilon, \lambda > 0$ there exists an integer $K = K(\varepsilon, \lambda)$ such that

$$F_{P_n z_n, Pz_n}(\frac{1-h}{4}\varepsilon) > 1 - \lambda$$

and

$$F_{T_n z_n, Tz_n}(\frac{1-h}{4}\varepsilon) > 1 - \lambda$$

for all $n \geq K$. so from (4.1) and (4.2) we have for all $n \geq K$

$$F_{z_n, z}(\varepsilon) > 1 - \lambda, \quad \text{since } F_{z_n, z}(\varepsilon) \leq F_{z_n, z}\left(\frac{1+2h}{2h}\varepsilon\right)$$

Thus $z_n \to z$.

THEOREM 5. Let $(X, \mathcal{F}, t)$ be a Menger space, where t is continuous and satisfies $t(x, x) \geq x$ for every $x \in [0, 1]$. Let a triplet $(P_n, Q_n : T_n)$ of self-mappings on X be a generalized contraction with (the same) generalized contraction constant h and z_n as their common fixed point for each $n = 1, 2, \cdots$. If $\{P_n\}$, $\{Q_n\}$ and $\{T_n\}$ converge respectively pointwise to self-mappings P, Q and T of X with z as their common fixed point, then $z_n \to z$.

PROOF. For any n,

$$\begin{aligned} F_{z_n, z}(\varepsilon) &= F_{P_n z_n, Qz}\left(\frac{1+h}{2}\varepsilon + \frac{1-h}{2}\varepsilon\right) \\ &\geq \min\left\{F_{P_n z_n, Q_n z}\left(\frac{1+h}{2}\varepsilon\right), \; F_{Q_n z, Qz}\left(\frac{1-h}{2}\varepsilon\right)\right\} \end{aligned}$$

(5.1) $$\geq \min\left\{F_{P_n z_n, Q_n z}\left(\frac{1+h}{2}\varepsilon\right), \; F_{Q_n z, Qz}\left(\frac{1-h}{4}\varepsilon\right)\right\}$$

Since $(P_n, Q_n : T_n)$ is a generalized contraction triplet, we have

$$\begin{aligned} F_{P_n z_n, Q_n z}\left(\frac{1+h}{2}\varepsilon\right) &= F_{P_n z_n, Q_n z}\left(h\frac{1+h}{2h}\varepsilon\right) \\ &\geq \min\left\{F_{T_n z_n, T_n z}\left(\frac{1+h}{2h}\varepsilon\right), \; F_{P_n z_n, T_n z_n}\left(\frac{1+h}{2h}\varepsilon\right),\right. \\ &\qquad \left. F_{Q_n z, T_n z}\left(\frac{1+h}{2h}\varepsilon\right), \; F_{P_n z_n, T_n z}\left(\frac{1+h}{h}\varepsilon\right), \; F_{Q_n z, T_n z_n}\left(\frac{1+h}{h}\varepsilon\right)\right\}, \end{aligned}$$

giving

$$\begin{aligned} F_{P_n z_n, Q_n z}\left(\frac{1+h}{2}\varepsilon\right) &\geq \min\left\{F_{T_n z_n, T_n z}\left(\frac{1+h}{2h}\varepsilon\right), \; F_{Q_n z, T_n z}\left(\frac{1+h}{2h}\varepsilon\right)\right\}, \\ &\qquad \text{since } P_n z_n = T_n z_n = z_n \\ &\geq \min\left\{F_{z_n, z}\left(\frac{1+2h}{2h}\varepsilon\right), \; F_{Tz, T_n z}\left(\frac{1-h}{2h}\varepsilon\right),\right. \\ &\qquad \left. F_{Q_n z, Qz}\left(\frac{1+h}{4h}\varepsilon\right), \; F_{Tz, T_n z}\left(\frac{1+h}{4h}\varepsilon\right)\right\} \end{aligned}$$

(5.2) $$\geq \min\left\{F_{z_n, z}\left(\frac{1+2h}{2h}\varepsilon\right), \; F_{Tz, T_n z}\left(\frac{1-h}{4}\varepsilon\right), \; F_{Q_n z, Qz}\left(\frac{1-h}{4}\varepsilon\right)\right\}.$$

Since Q and T are pointwise limits of $\{Q_n\}$ and $\{T_n\}$, for positive ε, λ corresponding to a point z, there exists an integer $K = K(\varepsilon, \lambda)$ such that

$$F_{T_n z, Tz}\left(\frac{1-h}{4}\varepsilon\right) > 1 - \lambda$$

and

$$F_{Q_n z, Qz}(\frac{1-h}{4}\varepsilon) > 1-\lambda$$

for all $n \geq K$.

so from (5.1) and (5.2) we have for all $n \geq K$,

$$F_{z_n, z}(\varepsilon) > 1-\lambda, \text{ since } F_{z_n, z}(\frac{1+2h}{2h}\varepsilon) \geq F_{z_n, z}(\varepsilon).$$

Hence $z_n \to z$.

REMARK 2. In Theorem 4, if $P=Q$ and T be an identity mapping on X then we obtain Ćirić's result [Theorem 2, 2] under a slightly different condition. In fact, in such a situation, Theorem 4 presents a slightly improved version of Ćirić's result [*op. cit*].

References

[1] Lj. B. Cirić, *Fixed and periodic points of almost contractive operators*, Math. Balkanica 3 (1973), 33-34.

[2] ______, *On fixed points of generalized contractions on probabilistic metric spaces*, Publ. Inst. (Beograd) (N. S.) 18 (32) (1975), 71-78.

[3] K. M. Das and K. Vishwanatha Naik, *Common fixed point theorems for commuting maps on a metric space*, Proc. Amer. Math. Soc. 77 (1979), 369-373.

[4] Brian Fisher, *Mappings with a common fixed point*, Math. Sem. Notes Kobe Univ. 7 (1979), 81-84.

[5] Olga Hadžić, *Some theorems on the fixed points in probabilistic metric and random normed spaces*, Boll. Un. Mat. Ital. B (5) 18 (1981).

[6] Vasile. I. Istrătescu, *Fixed point Theory*, D. Riedel Publishing Co., Holland, 1981.

[7] V. I. Istrătescu and I. Săcuiu, *Fixed point theorems on Probabilistic metric spaces*, Rev. Roumaine Math. Pures Appl. 9 (18) (1973), 1375-1380.

[8] D. S. Jaggi, *Fixed point theorems for orbitally continuous functions*, Mat. Vesnik 1 (14) (25) (1977), 129-135.

[9] Gerald Jungck, *Commuting mappings and fixed points*, Amer. Math. Monthly

83 (1976), 261 – 263.

[10] ______, *Periodic and fixed points and commuting mappings*, Proc. Amer. Math. Soc. 76 (1979), 333 – 338.

[11] S. Kasahara, *On some recent results on fixed points*, Math. Sem. Notes Kobe Univ. 6 (1978), 373 – 382.

[12] M. S. Khan, *Common fixed point theorems for a family of mappings*, Indian J. Pure Appl. Math. 12 (1981), 305 – 311.

[13] S. Park, *Fixed points of f-contractive maps*, Rocky Mountain J. Math. 8 (1978), 743 – 750.

[14] ______, *An extension of a fixed point thorem of Kasahara*, Math. Sem. Notes Kobe Univ. 7 (1979), 85 – 89.

[15] ______, *On general contractive type conditions*, J. Korean Math. Soc 17 (1980), 131 – 139.

[16] S. Park and J. S. Bae, *Extension of a fixed point theorem of Meir and Keeler*, Arkiv Mat. 19 (1981), 223 – 228.

[17] S. Park and B. E. Rhoades, *Extension of some fixed point theorems of Hegedus and Kasahara*, Math. Sem. Notes Kobe Univ. 9 (1981), 113 – 118.

[18] S. Ranganathan, *A fixed point theorem for commuting mappings*, Math. Sem. Notes Kobe Univ. 6 (1978), 351 – 357.

[19] B. Schweizer and A. Sklar, *Statistical metric spaces*, Pacific J. Math. 10 (1960), 313 – 334.

[20] V. M. Sehgal, *Some fixed point theorems in functional analysis and probability*, Ph. D. Dissertation, Wayne, state Univ., 1966.

[21] V. M. Sehgal and A. T. Bharucha-Reid, *Fixed points of contraction mappings on probabilistic metric spaces*, Math. Systems Theory 6 (1972), 97 – 102.

[22] H. Sherwood, *Complete probabilistic metric spaces*, Z. Wahrsch. Verw. Gebiete 20 (1971), 117 – 128.

[23] S. L. Singh, *On common fixed points of commuting mappings*, Math. Sem. Notes Kobe Univ. 5 (1977), 131 – 134.

[24] ______, *Application of common fixed point theorem*, Math. Sem. Notes Kobe Univ. 6 (1978), 37 – 40.

[25] ______, *A note on the convergence of a pair of sequences of mappings*, Arch. Math. (Brno), 15 (1979), 47 – 52.

[26] S. L. Singh and V. K. Gupta, ***On common fixed points of mappings***, J. Natur. Sci. Math. 19 (2) (1979), 239–245.

[27] S. L. Singh and C. Kulshrestha, ***Convergence of sequences of mappings and their common fixed points***, Math. Edu. (Siwan) 15 (1981), A, 55–60.

[28] S. L. Singh and B. D. Pant, ***Common fixed point theorems in probabilistic metric spaces***, (communicated).

[29] S. L. Singh and S. P. Singh, ***A fixed point theorem***, Indian J. Pure Appl. Math. 11 (1980), 1584–1586.

[30] B. M. L. Tiwari and S. L. Singh, ***Common fixed points of mappings in complete metric spaces***, Proc. Nat. Acad. Sci. India, Sect. A 51 (1981), 41–44.

[31] Cheh-Chin Yeh, ***On common fixed point theorems of continuous mappings***, Indian J. Pure Appl. Math. 10 (1979), 415–420.

[32] Chen Chih Yeh, ***On common fixed points of continuous mappings***, Math Sem. Notes Kobe Univ. 6 (1978), 115–121.

Department of Mathematics,
L. M. S. Government College,
Rishikesh, Dehradun 249 201
India

Department of Mathematics
Government College
Gopeshwar, Chamoli 246 401
India