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Decomposition in Noetherian Rings

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1. INTRODUCTION

The topic treated in this paper is an attempt to simulate in a noncommutative Noetherian ring the Lasker-Noether primary decomposition in a commutative Noetherian ring. For a large part, the results we shall give are due to Lesieur and Croisot.

A decomposition is given for an arbitrary left module as an intersection of special left modules -tertiary module- and we could hope that this would be a suitable vehicle for a profound study of noncommutative Noetherian rings and modules over noncommutative Noetherian rings.

All rings R to be considered will be Noetherian and will have unity. All modules M will be unital, finitely generated left R-modules. For any $m\in M$, $a\in R$ and submodule N of M, we denote by (m) the submodule of M generated by $m\in M$, by (a) the two-sided ideal of R generated by $a\in R$, by $(N:(a))=\{m\in M \mid (a)(m)\subseteq N\}$, and by $O(M)=\{a\in R \mid aM=(0)\}$. Note $(N:(a))$ is a submodule of M containing N.

2. TERTIARY RADICAL OF MODULES

DEFINITION 1. If N is a submodule of M, then the ***primary radical*** of N, written $rad(N)$, is the intersection of all prime ideals of R which contain $O(M/N)$. N is a ***primary submodule*** of M if all elements $a\in R$, such that for some submodule K of M with $N<K$ (not equal), $aK\subseteq N$, are in $\text{rad}(N)$.

DEFINITION 2. The ***tertiary radical*** $t\text{-}rad(N)$ of a submodule N of M (or an ideal N of a nonassociative ring $R=M$) is the set $t\text{-}rad(N)=\{a\in R \mid (N:(a))\cap(m)\subseteq N \Rightarrow m\in N, \text{ for all } m\in M\}$.

REMARK. For any ring, if $m\in \text{rad}(N)$, then $m^k\in O(M/N)$ for some integer k, since the intersection of all prime ideals is a nil ideal [(4., p. 56, Proposition 1)].

Thus if $m \in \mathrm{rad}(N)$, then $m^h \in \text{t-rad}(N)$ for some integer h.

PROPOSITION 1. (a). If N is a proper submodule of M, and $a \in \text{t-rad}(N)$, then $(N:(a)) \supset N$ (not equal).
(b). If A is an ideal of R, $A \subseteq \text{t-rad}(A)$.

Proof. (a). If $(N:(a)) = N$, and $a \in \text{t-rad}(N)$, then for all $m \in M$, $(N:(a)) \cap (m) = N \cap (m) \subseteq N$. Hence $m \in N$. Therefore $M = N$, a contradiction.
(b). Let $a \in A$. Then $(A:(a)) = R$. Hence for all $b \in R$, $(A:(a)) \cap (b) \subseteq A$ implies $(b) \subseteq A$ and so $b \in A$. Therefore $a \in \text{t-rad}(A)$.

LEMMA 2. Let N be a submodule of M. If $a_1, \cdots\cdots, a_n \in \text{t-rad}(N)$, then given $m \in M$, $m \notin N$, there is an $s \in (m)$, $s \notin N$ such that $a_i Rs \subseteq N$ for $i = 1, 2, \cdots, n$.

Proof. Let $a \in \text{t-rad}(N)$. Then by definition of $\text{t-rad}(N)$, for all $m \in M$, $m \notin N$, there exists $r \in (m)$, $r \notin N$ such that $aRr \subseteq N$. So if $n = 1$, this is obvious. Suppose then that we have found a $t \in (m)$, $t \notin N$ such that $a_i Rt \subseteq N$ for $i = 1, 2, \cdots\cdots, n-1$. Since $a_n \in \text{t-rad}(N)$, there is an $s \in (t) \subseteq (m)$, $s \notin N$ such that $a_n Rs \subseteq N$. However for $i < n$, $a_i Rs \subseteq a_i Rt \subseteq N$. Thereby the lemma is proved.

THEOREM 3. For any submodule N of M, $\text{t-rad}(N)$ is a twosided ideal of R.

Proof. For any $a_1, a_2 \in \text{t-rad}(N)$, by lemma 2, given $m \in M$, $m \notin N$, there is an $s \in (m)$, $s \notin N$ such that $a_1 Rs \subseteq N$, $a_2 Rs \subseteq N$. Thus $(a_1 - a_2) Rs \subseteq N$. Therefore, by definition of $\text{t-rad}(N)$, $a_1 - a_2 \in \text{t-rad}(N)$.
For any $r \in R$, $(ar) \subseteq (a)$ and $(ra) \subseteq (a)$. So $(N:(ar)) \supseteq (N:(a))$ and $(N:(ra)) \supseteq (N:(a))$. Thus if $a \in \text{t-rad}(N)$, since $(N:(ar)) \cap (m) \subseteq N$ implies $(N:(a)) \cap (m) \subseteq N$, then $ar \in \text{t-rad}(N)$, $ra \in \text{t-rad}(N)$.

COROLLARY 4. Let $I = \text{t-rad}(N)$, given $m \notin N$, there is an $s \in (m)$ such that $s \notin N$, $Is \subseteq N$.

Proof. Since R is Noetherian and I is an ideal of R, $I = Ra_1 + \cdots\cdots + Ra_n$ for some appropriate $a_i \in I$. By the lemma 2, pick $s \in (m)$, $s \notin N$ such that $a_i s \in N$ for $i = 1, 2, \cdots\cdots, n$. Thus $Is \subseteq N$.

COROLLARY 5. Let $I = \text{t-rad}(N)$ and let $N^* = \{ m \in M \mid Im \subseteq N \}$. Then N is a proper submodule of N^*.

Proof. Since I is a two-sided ideal of R, N^* is a submodule of M. Clearly $N^* \supseteq N$. By corollary 4, we can find $s \notin N$ such that $Is \subseteq N$. Since $s \in N^*$, $s \notin N$, N is a proper submodule of N^*.

LEMMA 6. Let N_1, N_2 be submodules of M. Then $\text{t-rad}(N_1) \cap \text{t-rad}(N_2) \subseteq \text{t-rad}(N_1 \cap N_2)$.

Proof. Let $a \in \text{t-rad}(N_1) \cap \text{t-rad}(N_2)$ and let $m \in M$ such that $(N_1 \cap N_2 : (a)) \cap (m) \subseteq N_1 \cap N_2$. Since $(N_1 \cap N_2 : (a)) = (N_1 : (a)) \cap (N_2 : (a))$, $N_1 \cap N_2 \subseteq N_2$, and $a \in \text{t-rad}(N_2)$, by definition of $\text{t-rad}(N_2)$ applied to $(N_1 : (a)) \cap (m)$, it follows that $(N_1 : (a)) \cap (m) \subseteq N_2$, so $(N_1 : (a)) \cap (m) \subseteq (N_2 : (a))$. Hence $(N_1 \cap N_2 : (a)) \cap (m) = (N_1 : (a)) \cap (N_2 : (a)) \cap (m) = (N_1 : (a)) \cap (m)$. Thus $(N_1 : (a)) \cap (m) \subseteq N_1 \cap N_2 \subseteq N_1$. Since $a \in \text{t-rad}(N_1)$, $m \in N_1$. Similarly $m \in N_2$. Therefore $m \in N_1 \cap N_2$, i.e. $a \in \text{t-rad}(N_1 \cap N_2)$.

3. TERTIARY MODULES

DEFINITION 3. Let N be a submodule of M. We call N a *tertiary submodule* of M if $(a)(m) \subseteq N$, $m \notin N \Rightarrow a \in \text{t-rad}(N)$.

DEFINITION 4. Let N be a tertiary submodule of M. If the prime ideal P of R is $\text{t-rad}(N)$, then we say that N is *P-tertiary* and that P is the *prime ideal of N*.

LEMMA 7. Let N be a tertiary submodule of M. Then $P = \text{t-rad}(N)$ is a prime ideal of R, i.e. N is $\text{t-rad}(N)$-tertiary.

Proof. Since $N^* \neq N$ and by lemma 2, there exists an $m \in M$, $m \notin N$ such that for all $a \in P$, $(a)(m) \subseteq N$. Since N is tertiary, P consists of all elements $a \in R$ such that $(a)(m) \subseteq N$. Suppose $x, y \in R$, $(x)(y) \subseteq P$, then $(x)(y)(m) \subseteq N$. If $y \notin P$ then $(y)(m) \nsubseteq N$. So by definition of $\text{t-rad}(N)$, $x \in P$. Therefore P is prime.

PROPOSITION 8. Let P be an ideal of R and N proper submodule of M. Suppose i) $P \subseteq \text{t-rad}(N)$, and

ii) $(a)(m) \subseteq N$, $m \notin N \Rightarrow a \in P$, for all $a \in R$, $m \in M$.

Then N is P-tertiary.

Proof. Since i) $P\subseteq$ t-rad(N) and ii) $(a)(m)\subseteq N$, $m\notin N\Rightarrow a\in P\subseteq t$-rad$(N)$, for all $a\in R$, $m\in M$, N is tertiary. For each $a\in$ t-rad(N), by proposition 1-(a), there exists $s\in (N: (a))-N$. Then $(a)(s)\subseteq N$ and $s\notin N$. Hence by ii) $a\in P$. Thus, with i), $P=$ t-rad(N).

PROPOSITION 9. Let N_1 and N_2 be P-tertiary submodules of M. Then $N_1\cap N_2$ is P-tertiary.

Proof. Let $N=N_1\cap N_2$. If $(a)(m)\subseteq N$, $m\notin N$, without loss of generality, $m\notin N_1$, then since $(a)(m)\subseteq N_1$, $a\in P=$ t-rad(N_1). By lemma 6, $P=$ t-rad(N_1) $\cap$ t-rad$(N_2)\subseteq$ t-rad$(N_1\cap N_2)=$ t-rad(N). By proposition 8, N is P-tertiary.

4. DECOMPOSITION OF MODULES

DEFINITION 5. A submodule N of M is called ***irreducible module*** if it is not the intersection of two strictly large submodules of M.

DEFINITION 6. A representation $N=N_1\cap\cdots\cdots\cap N_n$ of a submodule N of M as the intersection of tertiary submodules N_i of M is said to be ***irredundant*** if no N_i contains the intersection of N_j' s. $j\neq i$.

The representation is said to be ***reduced*** if it is irredundant and the t-rad(N_i) $\neq$ t-rad(N_j), $i\neq j$.

PROPOSITION 10. Every irreducible module is tertiary.

Proof. If N is not tertiary module, then there exists an $a\notin$ t-rad(N), $m\notin N$ such that $(a)(m)\subseteq N$. Since $a\notin$ t-rad(N), there is a $t\notin N$ with $(N: (a))\cap (t)\subseteq N$. Since $N\subseteq (N: (a))$, and the lattices of modules of M is modular, $N=(N: (a))\cap ((t)+N)$. But $(a)(m)\subseteq N$, $m\notin N$, so $N\subsetneq (N: (a))$, and since $t\notin N$, $N\subsetneq (t)+N$. Therefore N is not irreducible.

LEMMA 11. Every submodule of M is the intersection of finitely many irreducible modules.

Proof. Let $\mathscr{F}$ be the family of all submodules of M which are not expressible as the intersection of finitely many irreducible modules. Suppose $\mathscr{F}$ is not empty. Then there exists a maximal element N of $\mathscr{F}$. Then $N=A\cap B$ for some submodules A, B of M such that $N \subsetneq A$ and $N \subsetneq B$. So $A \notin \mathcal{F}$ and $B \notin \mathcal{F}$. Therefore A and B are expressible as the finitely many irreducible modules. It follows that N is expressible as the finitely many irreducible modules, i.e. $N \notin \mathcal{F}$, a contradiction.

COROLLARY 12. Every submodule of M has the reduced representation.

Proof. By lemma 11, proposition 10, and proposition 9.

LEMMA 13. For any submodule N, $\mathrm{rad}(N) \subseteq \text{t-rad}(N)$.

Proof. By the earlier remark, if $x \in \mathrm{rad}(N)$ there exists an integer $k(x)$ such that $x^{k(x)} \in \text{t-rad}(N)$. Since R is Noetherian by (2, p. 90 Theorem 5.1), $(\mathrm{rad}(N))^h \subseteq \text{t-rad}(N)$, for some integer h. By the definition of t-rad(N), $\mathrm{rad}(N) \subseteq \text{t-rad}(N)$.

THEOREM 14. If R is a commutative Noetherian ring and if N is a submodule of M, then $\mathrm{rad}(N) = \text{t-rad}(N)$.

Proof. Let $a \in \text{t-rad}(N)$ and let $J_n = (N : (a^n))$. Then J_n form an ascending chain of submodules of M, hence for some k, $J_k = J_{k+1}$. If $a^k M \subseteq N$, we can find $m \in M$ with $a_k m \notin N$. Since $a \in \text{t-rad}(N)$ there is an $r \in R$ with $ra^k m \notin N$ but for $(a)(ra^k m) \subseteq N$. Since R is commutative, this yields $(a^{k+1})(rm) \subseteq N$, that is, $rm \in J_{k+1} = J_k$, hence $a^k rm \in N$, contrary to $a^k rm \notin N$. Thus $a \in \text{t-rad}(N)$ implies $a^k M \subseteq N$ for some integer k, so $a^k \in \mathrm{rad}(N)$. Since $\mathrm{rad}(N)$ is the intersection of prime ideals, $a \in \mathrm{rad}(N)$, that is $\text{t-rad}(N) \subseteq \mathrm{rad}(N)$. Therefore $\text{t-rad}(N) = \mathrm{rad}(N)$.

LEMMA 15. If N, U, U', V V' are submodules of M such that $N = U \cap V = U' \cap V'$, U is P-tertiary, U' is P'-tertiary and $P \neq P'$. Then $N = V \cap V'$.

Proof. Let $m \in V \cap V'$. Since $P \neq P'$, without loss of generality, there exists an $a \in P$, $a \notin P'$. If $m \notin N$, there exists a $t \in (m)$, $t \notin N$ such that $(a)(t) \subseteq N$. Thus $(a)(t) \subseteq U' \cap V'$. Since U' is P'-tertiary and $(a)(t) \subseteq U'$, $a \in \text{t-rad}(N') = P'$, a contradiction. Thus $m \in N$, that is $V \cap V' \subseteq N$. Since $N \subseteq V$, $N \subseteq V'$, we have $N = V \cap V'$.

THEOREM 16. If the submodule N of M has the two reduced representations $N = N_1 \cap \cdots\cdots \cap N_r = N'_1 \cap \cdots\cdots \cap N'_s$, then $r = s$, and the prime ideals $P_i = \text{t-rad}(N_i)$ coincide with the prime ideals $P'_i = t\text{-rad}(N'_i)$ after renumbering.

Proof. $P_1 = P'_i$ for some i. If not, since $P_1 \neq P'_1$, by lemma 15, $N = N_2 \cap \cdots\cdots \cap N_r \cap N'_2 \cap \cdots\cdots \cap N'_s$. Since $P_1 \neq P'_2$, and $N = N_1 \cap \cdots\cdots \cap N_r = N_2 \cap \cdots\cdots \cap N_r \cap N'_2 \cap \cdots\cdots \cap N'_s$, $N = N_2 \cap \cdots\cdots \cap N_r \cap N'_3 \cap \cdots\cdots \cap N'_s$. Continuing we arrive at $N = N_2 \cap \cdots\cdots \cap N_r$ contrary to the irredundanty of the representation $N = N_1 \cap \cdots\cdots \cap N_r$. Thus $P_1 = P_i'$ for some i. In the same way, given j, $P_j = P'_k$ for some k. This shows $r \leqq s$. This is symmetric, so $s \leqq r$. Thus $r = s$, and $\{P_i\}_{i=1,2,\cdots,r} = \{P'_j\}_{j=1,2,\cdots,s=r}$.

LEMMA 17. A P-tertiary submodule N of M is primary iff $P^n M \subseteq N$ for some n.

Proof. If N is primary, then $P = \text{rad}(N)$. Since $x \in \text{rad}(N)$ implies $x^{k(x)} \in O(M/N)$ So $P^n \subseteq O(M/N)$ ([2, p. 90 Theorem 5.1.]). Hence $P^n M \subseteq N$. On the other hand, if $P^n M \subseteq N$, $P^n \subseteq \text{rad}(N)$. Since $\text{rad}(N)$ is the intersection of prime ideals, this puts P in $\text{rad}(N)$. Together with $\text{rad}(N) \subseteq P$, $P = \text{rad}(N) = \text{t-rad}(N)$.

References

1. E. A. Behrens, *Ring Theory*, Academic Press, New York and London, 1972, pp. 249~259.
2. I. N. Herstein, *Topics in Ring Theory*, The University of Chicago Press, 1969, pp. 87~114.
3. T. W. Hungerford, *Algebra*, Holt, Rinehart and Winston, Inc., 1974, pp. 377~394.
4. J. Lambek, *Lectures on Rings and Modules*, Blaisdell Pudlishing Company, 1966, pp. 101~106.
5. D. G. Northcott, *Ideal Theory*, Cambridge University Press, 1972, pp. 13~30.
6. J. A. Riley, *Axiomatic Primary and Tertiary Decomposition Theory*, Trans. Amer. Math. Soc., 105(1962), pp. 177~201.