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Fiber Spaces and Bundle Properties

By Dae-Shik Chun

1. Introduction

There are two notions of a fiber space and a fibre bundle, and each has several different notions : Hurewicz fiber space, Serre fiber space etc and principal bundle, fibre bundle etc. There are fiber spaces that are not fibre bundle, and a fiber bundle is not necessarily a fiber space. The object of the definition of fiber spaces is to state a minimum conditions under which the covering homotopy property holds, and the leading condition of the defintion of fibre bundles is the locally trivial condition.

We know that under some condition, a fiber structure with the SSP has the covering homotopy property (cf, [2], p.405), and any fibre bundle has the SSP (cf, [7], p.760), that is, a fiber structure with the SSP performs the intermediate role of the fiber spaces and the fibre bundles.

In this paper, we shall give a condition in order that a fiber structure with the SSP has the locally trivial condition (Theorem 1), and we shall give a property of the fibre bundles (Theorem 3).

2. Definitions

In this section, we recall various definitions.

(A) A fiber structure is a triple (E, p, B) consisting of two spaces E, B and a continuous surjection $p : E \longrightarrow B$

(B) A fiber structure (E, p, B) is said to have the covering homotopy property (CHP) for the space X if, for every map $f : X \longrightarrow E$ and every homotopy $g_t : X \longrightarrow B$ of the map $pof (g_0 = pof)$, there exists a homotopy $f_t : X \longrightarrow E$ of f such that $pof_t = g_t$.

(C) A fiber structure (E, p, B) is called a fiber space for a class $\mathscr{A}$ of spaces if it has the CHP for each $X \in \mathscr{A}$.

(D) A fiber structure (E, p, B) is called a Hurewicz fiber space if it has the CHP for every space X.

(E) A fiber structure (E, p, B) is called a Serre fiber space if it has the CHP for every triangulable space X.

(F) A fiber structure (E, p, B) is said to have the slicing structure proterty (SSP) if for each $b \in B$ there is an open neighborhood U of b and a continuous function $\phi_U : U \times p^{-1}(U) \longrightarrow E$ such that

$$(1)\ p \cdot \phi_U(b, x) = b \qquad \text{for all} \quad (b, x) \in U \times p^{-1}(U),$$

$$(2)\ \phi_U(p(x), x) = x \qquad \text{for each } x \in p^{-1}(U).$$

The maps ϕ_U are called slicing functions and the neighborhoods U are called slicing neighborhoods. The fiber structure (E, p, B) with the SSP is also called the fiber space in the sense of Huebsch and Hu (cf. [8]).

(G) Let $\xi = (E, p, B)$ and $\xi' = (E', p', B')$ be two fiber spaces for a class $\mathscr{A}$ of spaces, and let $\psi : E \longrightarrow E'$ be a continuous map. ψ is called a fiber map of ξ into ξ', if there exists a continuous map $f : B \longrightarrow B'$ such that $p' \cdot \psi = f \cdot p$. f is called the induced map by ψ.

Let $\xi = (E, p, B)$ and $\xi' = (E', p', B)$ be two fiber spaces over B for a class $\mathscr{A}$ of spaces. A fiber map $\psi : E \longrightarrow E'$ is called equivalent map and ξ and ξ' are said to be equivalent, if the following two conditions are satisfied:

i) ψ maps E onto E' homeomorphically.

ii) The induced map $f : B \longrightarrow B$ is the identity map.

(H) Let $\xi = (E, p, B)$ be a fiber space for a class $\mathscr{A}$ of spaces. For any space F and the projection $p : B \times F \longrightarrow B$ onto B, $(B \times F, p, B)$ is a fiber space for a class $\mathscr{A}$ of spaces. If ξ and $(B \times F, p, B)$ are equivalent, ξ is called a trivial fiber space with fiber F.

A fiber structure (E, p, B) is said to have the locally trivial condition if for each $b \in B$, there exists an open neighborhood U of b such that $(p^{-1}(U), p \mid p^{-1}(U), U)$ is equivalent to $(U \times F, p, U)$, that is, $(p^{-1}(U), p \mid p^{-1}(U), U)$ is trivial fiber space with fiber F.

(I) A fiber structure (E, p, B) is called a locally trivial fiber space, if there exists a space F such that, for each $b \in B$, there is an open neighborhood U of b together with a homeomorphism

$$\theta_U : U \times F \longrightarrow p^{-1}(U)$$

of $U\times F$ onto $p^{-1}(U)$ satisfying the condition:

$$p\cdot\theta_U(u, y) = u \text{ for all } (u, y) \in U\times F,$$

that is, $(p^{-1}(U), p|p^{-1}(U), U)$ is a trivial fiber space with fiber F. The open sets U and the θ_U will be called the coordinate neighborhoods and the coordinate functions respectively. The locally trivial fiber space (E, p, B) is also said to have the bundle property (cf. [6] p. 65).

For a fiber structure (E, p, B) and a space F, $\xi=(E, p, B)$ is a locally trivial fiber space if and only if there exist an open covering $\{U_\lambda\}_{\lambda\in\Lambda}$ of B and coordinate functions $\theta_\lambda: U_\lambda\times F \longrightarrow p^{-1}(U_\lambda)$. These system $\{U_\lambda, \theta_\lambda\}_{\lambda\in\Lambda}$ is called coordinate neighborhood system of ξ.

(J) Let (E, p, B) be a fiber structure, and let G be a topological transformation group acting on E on the right. Then $\eta=(E, p, B, G)$ is called a principal bundle having G as a structure group if, for each $b\in B$, there is an open nighborhood U of b together with an onto homeomorphism

$$\phi: U\times G \longrightarrow p^{-1}(U)$$

satisfying the conditions:

i) $p\cdot\phi(b, g) = b$

ii) $\phi(b, g)\cdot g' = \phi(b, gg')$ for all $b\in U$; $g, g'\in G$.

By the definition, a principal bundle (E, p, B, G) with G as fiber is a locally trivial fiber space.

3. Main Theorems.

First of all, before entering our main theorems, we shall introduce the some well known results without proofs, which describe the inter-relations among the above various notions of fiber structure.

Proposition 1. Let (E, p, B) be a fiber structure with the SSP. Then

(1) (E, p, B) is a fiber space for the paracompact spaces.

(2) If B is paracompact, then (E, p, B) is a Hurewicz fiber space (cf. [2], p. 405).

Proposition 2. Let B be paracompact and locally equi-connected. Then a fiber structure (E, p, B) is a Hurewicz fiber space if and only if it has the SSP. (cf. [2]. p. 405).

Proposition 3. Let (E, p, B, F) be a locally trivial fiber space. Then the fiber structure (E, p, B) has the CHP for the class of paracompact Hausdorff spaces. Since I^n is a paracompact Hausdorff space, (E, p, B) is a fiber space in the sense of Serre. ([13]. p. 168).

Now we shall give a condition in order that a fiber structure with the SSP has the locally trivial condition:

THEOREM 1. Let (E, p, B) be a fiber structure with the SSP.
If i) for any two $a, b \in B$, $p^{-1}(a)$ and $p^{-1}(b)$ are homeomorphic, and the homeomorphism is denoted by $g_{ab}: p^{-1}(a) \longrightarrow p^{-1}(b)$ and ii) the slicing function ϕ_U is a bijection and satisfies the condition:

$$g_{ba}\ \phi_U\ (b, x) = x \text{ for } a, b \in U,\ x \in p^{-1}(a),$$

then (E, p, B) has the locally trivial condition, where $\{U\}$ is the slicing neighborhood system and $U \in \{U\}$.

(Proof). Let F be any topological space homeomorphic with $p^{-1}(b)$ for each $b \in B$, and let $f_u: F \longrightarrow p^{-1}(b_0)$ be a homeomorphism for $U \in \{U\}$ and $b_0 \in U$ (by the given condition i), such a homeomorphism exists).
Lets define $\theta_u: U \times F \longrightarrow p^{-1}(U)$
by taking $\theta_u(b, y) = \phi_u(b, f_u(y))$, for each $(b, y) \in U \times F$. Since ϕ_u and f_u are bijective, ϕ_u is bijective, and since ϕ_u and f_u are continuous, θ_u is continuous. By the property of the slicing function ϕ_u, we have

$$p\ \theta_u(b, y) = p\ \phi_u(b, y_u(y)) = b$$

Now lets define $\psi_u: p^{-1}(U) \longrightarrow U \times F$
by taking $\psi_u(x) = (p(x), f_u^{-1}\ g_{p(x)\, b0}\ (x))$.
Since p, f_u^{-1} and $g_{p(x), b0}$ are continuous, ψ_u is continuous. Now we show that ψ_u is an inverse of θ_u.

$$\begin{aligned}\psi_u \cdot \theta_u(b, y) &= (p(\theta_u(b, y)), f_u^{-1}\ g_{p(\theta_u(b,y)), b0}\ (\theta_u(b, y)))\\ &= (b, f_u^{-1} \cdot g_{b, b0}\ (\theta_u(b, y)))\\ &= (b, f_u^{-1} \cdot g_{b, b0}\ \phi_u(b, f_u(y)))\\ &= (b, f_u^{-1} \cdot f_u(y)) \qquad \text{(by the given condition ii))}\\ &= (b, d)\end{aligned}$$

Similarly, we can show that $\theta_u \cdot \psi_u = 1_{p^{-1}(u)}$ holds. Thus θ_u is a homeomorphism and (E, p, B) has the locally trivial condition.

Here we shall recall the definitions of a fibre bundle and the related notions.

Locally trivial fiber space (E, p, B, F), its coordinate neighborhood system $\{U_\lambda, \theta_\lambda\}$ $\lambda \in \Lambda$ and a transformation group G acting on F on the left are given. If there exists a family of continuous maps $g_{\mu\lambda}: U_\lambda \cap U_\mu \to G$ $(\lambda, \mu \in \Lambda)$ satisfying the condition :

$$\theta_\lambda(b, y) = \theta_\mu(b, g_{\mu\lambda}(b) \cdot y) \text{ for } b \in U_\lambda \cap U_\mu,\ y \in F,$$

$(E, P, B, F, G, \{U_\lambda, \theta_\lambda\})$ is called a coordinate bundle, and E, P, B, F, G and $\{U_\lambda, \theta_\lambda\}$ are called total space, projection, base space, fiber, structure group and coordinate neighborhood system respectively, and $\{g_{\mu\lambda}\}$ is called coordinate translation system. Let $(E, P, B, F, G, \{U_\lambda, \theta_\lambda\})$ and $(E, P, B, F, G, \{V_\alpha, \psi_\alpha\})$ be the two coordinate bundles with the same total space, projection, base space, fiber and structure group. If there exists a family of continuous maps $h_{\alpha\lambda}: U_\lambda \cap V_\alpha \to G$ satisfying the condition :

$$\theta_\lambda(b, y) = \psi_\alpha(b, h_{\alpha\lambda}(b) \cdot y),\quad b \in U_\lambda \cap V_\alpha,\ y \in F,$$

they are said to be equivalent. This is an equivalent relation. The equivalent class is called a fibre bundle, and we denote it by $\xi = (E, P, B, F, G)$ (cf. [7], p. 759 and [13], p. 200).

Proposition 4. Let $\xi = (E, p, B, G)$ be a principal bundle, and let F be a space such that G acts on F on the left. Then G acts on $E \times F$ on the right by the relation

$$(x, y) \cdot g = (xg, g^{-1}y) \cdot$$

Let E_F denote the orbit space $E \times F/G$, and $p_F: E_F \to B$ be defined by taking $p_F[(x, y)G] = p(x)$ for $(x, y) \in E \times F$ and $(x, y)G \in E \times F/G$. Then (E_F, p_F, B, F, G) denoted $\xi[F]$ is a fibre bundle with fiber F and structure group G which is called an associated fibre bundle of ξ (cf. [13], p. 201 and [12], p. 44).

Now in order to give a property of the fibre bundle, we shall consider a following lemma :

Lemma 2. Let B be a given space and let G be a topological group. If a system of continuous maps $\{g_{\mu\lambda}\}$ on B taking values in G $(g_{\mu\lambda}: U_\lambda \cap U_\mu \to G)$ is given, then a principal bundle (E, P, B, G) is determined, where $\{U_\lambda\}_{\lambda \in \Lambda}$ is an open covering of B.

(Proof) We can view G as a transformation group on G itself on the left. Consider a fiber space (E, p, B, G) with G as fiber. Let $h: \bigcup_{\lambda \in \Lambda} U_\lambda \times G \times \lambda \to E$ be the identification map. Then G acts on E on the right by the relation $h(b, g,$

$\lambda) g' = h(b, gg', \lambda)$, for $b \in U_\lambda$, g, $g' \in G$, $\lambda \in \Lambda$.

Define a map $\phi_\lambda : U_\lambda \times G \to p^{-1}(U_\lambda)$ by taking

$$\phi_\lambda(b, g) = h(b, g, \lambda) \text{ for } b \in U_\lambda, \ g \in G.$$

Then $(E, p, B, G,)$ is a principal bundle with $\{U_\lambda, \phi_\lambda\}$ as the system of coordinate neighborhoods (cf. [13] p. 185).

THEOREM 3. Let B and B' be the two given spaces which are different, and let G be a given topological group. If two systems of continuous maps $\{g_{\mu\lambda}\}$ on B and $\{g'_{\mu\lambda}\}$ on B' both taking values in G, then two fibre bundles with the same total space, the same structure group G and having the each given space B and B' as the base spaces are always determined.

(Proof). By the lemma 2, two principal bundles $\xi = (E, p, B, G)$ and $\xi' = (E', p', B, G)$ are determined. By giving a relation

$$g \cdot e' = e' \cdot g^{-1},$$

we can view that G acts on E' on the left. Thus we get an associated fibre bundle $\xi[E'] = (E_{E'}, P_{E'}, B, G)$ (see Proposition 4). In the same way, we get an other associated fibre bundle $\xi'[E] = (E'_E, p'_E, B, G)$.

We have :

$$E_{E'} = E \times E'/G = E' \times E/G = E'_E$$

Therefore $\xi[E']$ and $\xi'[E]$ are two fibre bundles with the same total space.

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