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Holonomy Groups and Distributions on a Riemannian Manifold

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§ 1. Introduction

Recently, Riemannian geometry has been connected with topology, and it is greatly indebted to manifold theory that is dazedly developed ([3], [5]). We have known that holonomy groups, distributions and curvature tensors are connected to each others ([7], [9]). The purpose of this paper is to prove a relation between distributions and holonomy groups on a Riemannian manifold (Theorem 5, 5) which is deeply concerned to the curvature tensor K.

An outline of this paper is described as follows.

§ 2 is a preparation for § 3, § 4 and § 5. Accordingly, in § 2 notations and fundamental concepts which shall be used in this paper are provided. In § 3, a distribution on a Riemannian manifold is defined, and several properties about distributions are proved (Propositions 3.4, 3.6, 3.7, 3.8, 3.9 and 3.10). In § 4 we shall describe a definition of a holonomy group on a Riemannian manifold and prove some properties with respect to holonomy groups (Proposition 4.4, 4.5 and 4.8). § 3 and § 4 are of course a preparation for § 5. § 5 is our main part, and our main theorem (Theorem 5.5) shall be proved in this section.

Throughout this paper, by a manifold we mean a smooth manifold without boundary.

§ 2. Preliminaries

Let (M, g) be an n-dimensional Riemannian manifold without boundary over R (reals), where g is a Riemannian metric of M which has g_{ji} as its components in a coordinate neighborhood $(U : X^1, \cdots\cdots, X^n)$. Let X and Y be (C^∞) vector fields (i, e., $X, Y \in *(M)$), and let us assume that $X = X^h \frac{\partial}{\partial x^h}$ and $Y = Y^h \frac{\partial}{\partial x^h}$ on U. Then we have

$$\langle X, Y\rangle = g(X, Y) = g_{ji} X^j X^i,$$

which is called the ***inner product*** of X and Y on U. Suppose the $n\times n$- matrix (g_{ji}) and put

$$(g^{ji})=(g_{ji})^{-1}$$

Then, we get the symmetric tensor field of type (2, 0)

$$G_U=g^{ji}\frac{\partial}{\partial x^j}\otimes\frac{\partial}{\partial x^i}$$

in U. There is a symmetric tensor field G of type (2, 0) such that for each U $G\mid U=G_U\cdot g^{ji}$ is called a ***contravariant component*** of g on $(U:X^h)$. The following hold :

$$(g_{ji})\ (g^{ki})=(\delta_j^k),\quad g_{ji}g^{ki}=\delta_j^k.$$

Definition 2.1 Let M be a (C^∞) manifold, and let $*(M)$ be the set of all vector fields. A map $\nabla: *(M)\times *(M)\rightarrow *(M)$ is called a ***pseudo connection*** on M if it satisfies the following conditions.

Put $\nabla_X Y=\nabla(X,Y)$ for $X, Y\in *(M)$.

(i) $\nabla_X(Y+Z)=\nabla_X Y+\nabla_X Z,\ Z\in *(M)$

(ii) $\nabla_X(fY)=f\nabla_X Y+(Xf)Y,\ f: M\rightarrow R: C^\infty$ class

(iii) $\nabla_{X+Y}\ Z=\nabla_X Z+\nabla_Y Z$

(iv) $\nabla_{fX}Y=f\nabla_X Y$

It is easily proved that

(i) $X=0\Rightarrow V_XY=0,\ Y=0\Rightarrow V_XY=0\ (X\in Y*(M))$

(ii) $\nabla_{aX}Y=a\nabla_X Y,\ \nabla_X(aY)=a\nabla_X Y(X, Y\in *(M),\ a\in R)$

Definition 2.2 Let ∇ be a pseudo connection on M, and let $(U:X^h)$ be a coordinate neighborhood of M. In U, we put

$$e_1=\frac{\partial}{\partial x^1},\ \cdots\cdots\cdots\cdots,\ e_n=\frac{\partial}{\partial x^n}$$

Then, we can write as

$$\nabla_{ej}^{ei}=\Gamma_{ji}^h e_h$$

The C^∞ function $\Gamma_{ji}^h\ (i,j,h=1,2,\cdots\cdots,n)$ are called the ***connection coefficients*** of ∇ on U.

Definition 2.3. We shall denote $\mathcal{J}_s^r$ the set of all tensor fields of type (r,s). Then, for $X\in *(M)$ $\nabla_X: \mathcal{J}_s^r\rightarrow\mathcal{J}_s^r$ is defined as follows.

(i) Put $\mathcal{J}_0^0=C^\infty(M)$ which is the set of all C^∞ functions defined on M. We define

$$\begin{array}{ccc} \nabla_X : \mathfrak{T}^0_0 & \longrightarrow & \mathfrak{T}^0_0 \\ \Psi & & \Psi \\ f & \longmapsto & \nabla_X f = Xf \end{array}$$

(ii) For $u \in \mathfrak{T}^1_0$ we define $\nabla_X u$ by

$$(\nabla_X u\ (Y) = \nabla_a (u(Y)) - u(\nabla_X Y),\quad Y \in * (M)$$

(iii) $\nabla_X(S+T) = \nabla_X S + \nabla_X T \quad (S, T \in \mathfrak{T}^p_q)$

$$\nabla_X(S \otimes P) = \nabla_X S \otimes P + S \otimes \nabla_X P \quad (P \in \mathfrak{T}^r_s)$$

By induction and (iii) $\nabla_X : \mathfrak{T}^r_s \longrightarrow \mathfrak{T}^r_s$ is defined, and we call ∇_X the ***covariant differential*** with respect to X. We also define

$$\nabla : \mathfrak{T}^r_s \longrightarrow \mathfrak{T}^r_{s+1}$$

by $\nabla S(u_1, \cdots, u_r,\ X, X_1, \cdots, X_s) = \nabla_X S(u_1, \cdots, u_r,\ X_1, \cdots, X_r)$. ∇S is called the ***covariant derivative*** of S.

Proposition 2.4. Let X^h be a component of $X(\in * (M)$ in $(U : X^h)$. Then $\nabla_X dX^h = -X^j \Gamma^h_{ji} dX^i$

Proof. Let Y^h be a component of $Y(\in * (M))$ in $(U : X^h)$. Then

$$(\nabla_X dx^h)(Y) = \nabla_X(Y^h) - dx^h (X^j \frac{\partial r^i}{\partial x^j} \frac{\partial}{\partial x^i} + X^j Y^i \ \Gamma^h_{ji} \frac{\partial}{\partial x^h}$$

$$= X^j \frac{\partial Y^h}{\partial x^j} - X^j \frac{\partial Y^h}{\partial x^j} - X^j Y^i \ \Gamma^h_{ji}$$

$$= -X^j Y^i \ \Gamma^h_{ji}$$

$$(-X^j \Gamma^h_{ji} dx^i)\ \ (Y) = -X^j Y^i \Gamma^h_{ji}$$

$$\therefore \nabla_X dx^h = -X^j \Gamma^h_{ji} dx^i$$

q. e. d.

Definition 2.5. Let ∇ be a pseudo-connection of Riemannian manifold (M, g). ∇ is called the ***Riemannian connection*** of M if it satisfies the following conditions :

(i) ∇ is symmetric (*i. e.*, $\Gamma^h_{ji} = \Gamma^h_{ji}$)

(ii) $\nabla g = 0$

Sometimes ∇ is called the Levi-Civita's connection.

We see that

$$\nabla g = 0 \Longleftrightarrow \nabla_X \langle Y, Z\rangle + \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle\ (X, Y, Z \in * (M)).$$

Proposition 2.6. For the Riemannian connection ∇ of M

$$\Gamma^h_{ji} = \{^h_{ji}\} = \frac{1}{2} g^{hk} (\frac{\partial g_{ik}}{\partial x^j} + \frac{\partial g_{jk}}{\partial x^i} - \frac{\partial g_{ji}}{\partial x^k})$$

where $\{^h_{ji}\}$ =is a ***Christoffel' s symbol*** .

Proof. From (ii) we have

$$\frac{\partial g^{ji}}{\partial x^k} - \Gamma^m_{kj} g_{mi} - \Gamma^m_{ki} g_{jm} = 0$$

Put $\Gamma_{ji,h} = \Gamma^m_{ji} g_{mi}$ then

$$\Gamma_{ji,h} = \Gamma_{ij,h} \quad (1) \qquad \Gamma_{kj,i} + \Gamma_{ki,j} = \frac{\partial g_{ji}}{\partial x^k} \quad (2)$$

We replace i, j, k with j, k and i in regular sequence twice, then we get

$$\Gamma_{ik,j} + \Gamma_{ij,k} = \frac{\partial g_{kj}}{\partial x} \quad (3) \qquad \Gamma_{ji,k} + \Gamma_{jk,i} = \frac{\partial g_{ik}}{\partial x^j} \quad (4)$$

(3)+(4)−(2) and using (1) we have the following

$$\Gamma_{ji,k} = \frac{1}{2}\left(\frac{\partial g_{ik}}{\partial x^j} + \frac{\partial g_{jk}}{\partial x^i} - \frac{\partial g_{ji}}{\partial x^k}\right)$$

$$\therefore \Gamma^m_{ji} = \frac{1}{2} g^{hk}\left(\frac{\partial g_{ik}}{\partial x_j} + \frac{\partial g_{jk}}{\partial x_i} - \frac{\partial g_{ji}}{\partial x_k}\right). \qquad q.e.d.$$

As is well know, every Riemannian manifold (M, g) has only one its Riemannian connection, and each connection coefficient of ∇ is given by Christoffel's symbol $\{^h_{ji}\}$. In particular, for $X, Y \in *(M)$

$$[X, Y] = \nabla_X Y - \nabla_Y X \qquad ([3])$$

Definition 2.7. Let ∇ be the Riemannian connection of an n-dimensional Riemannian manifold (M, g). A map $K: *(M) \times *(M) \times *(M) \to *(M)$ is defined as follows. For $X, Y, Z \in *(M)$

$$K(X, Y, Z) = K(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$

In this case, K is called the ***curvature tensor*** of (M, g) or the ***Riemann Christoffl*** tensor of (M, g). We can prove that

$$K(fX, gY)(hZ) = (fgh) K(X, Y) Z \; (f, g, h \in C^\infty(M),\ X, Y, Z \in *(M))$$

([3]).

Let $(U: x^h)$ be a coordinate neighborhood of (M, g). The components of the Riemannian connection ∇ of M are $\{^h_{ji}\}$ $(1 \leq h, i, j \leq n)$. We put

$$e_1 = \frac{\partial}{\partial x^1} \cdots\cdots\cdots e_n = \frac{\partial}{\partial x_n}$$

in U,

$$K(e_k, e_j) = K_{kji}{}^h e_h$$

Proposition 2.8. $K_{kji}{}^h = \frac{\partial}{\partial x^k}\{^h_{ji}\} - \frac{\partial}{\partial x^j}\{^h_{ki}\} + \{^h_{km}\}\{^m_{ji}\} - \{^h_{jm}\}\{^m_{ki}\}$

Proof. $K(e_k, e_j)(e_i)$

$$= \nabla_{ek}\nabla_{ej} e_i - \nabla_{ej}\nabla_{ek} e_i - \nabla_{[ek, ej]} e_i$$

$$= \nabla_{ek}(\{^h_{ji}\} e_h) - \nabla_{ej}(\{^h_{ki}\} e_h) - \nabla_{[\nabla_{e_k} e_j - \nabla_{e_j} e_k]} e_i$$

$$= \left\{{h \atop ji}\right\}\left\{{m \atop kh}\right\} e_m + \frac{\partial}{\partial x^k}\left\{{h \atop ji}\right\} e_h - \left\{{h \atop ki}\right\}\left\{{m \atop jh}\right\} e_m - \frac{\partial}{\partial x^j}\left\{{h \atop ki}\right\} e_h \; (\nabla_{e_k} e_j = \nabla_{e_j} e_k)$$

$$= \left(\frac{\partial}{\partial x^k}\left\{{h \atop ji}\right\} - \frac{\partial}{\partial x^j}\left\{{h \atop ki}\right\} + \left\{{m \atop ji}\right\}\left\{{h \atop km}\right\} - \left\{{m \atop ki}\right\}\left\{{h \atop jm}\right\}\right) e_h$$

$$= K_{kji}{}^{h} e_h .$$

$$\therefore K_{kji}{}^{h} = \frac{\partial}{\partial x^k}\left\{{h \atop ji}\right\} - \frac{\partial}{\partial x^j}\left\{{h \atop ki}\right\} + \left\{{h \atop km}\right\}\left\{{m \atop ji}\right\} - \left\{{h \atop jm}\right\}\left\{{m \atop ki}\right\} . \qquad q.\ e.\ d.$$

The following properties with respect to K have already proved.

(i) $K_{kji}{}^{h} = -K_{jki}{}^{h}$

(ii) $K_{kji}{}^{h} + K_{jik}{}^{h} + K_{ikj}{}^{h} = 0$

Definition 2. 9. Given two vector fields $X, Y \in *(M)$ we define a map $K(X, Y) : *(M) \to *(M)$ by

$$K(X, Y)(Z) = K(X, Y) Z$$

for $Z \in *(M)$. It follows that

$$K(X, Y)(fZ) = fK(X, Y)\ (Z) \quad (f \in C^{\infty}(M),\ Z \in *(M).$$

$K(X, Y)$ is a tensor field of type $(1, 1)$. and which is called a ***curvature transformation.*** If X^h and Y^h are components of X and Y in a coordinate neighborhood $(U : X^h)$, respectively, then the components of the curvature transformation $K(X, Y)$ are

$$K(X, Y)_i^h = X^h Y^j K_{kji}{}^{h} \quad (i, h = 1, 2, \cdots, n)$$

§ 3. Distributions

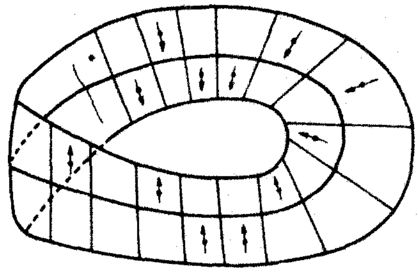
Let M be an n-dimensional smooth manifold.

Definition 3. 1 A distribution D of dimension r on M is a map

$$\begin{array}{lcl} D : M & \longrightarrow & T(M) \\ \quad \Psi & & \\ \quad p & \longmapsto & D(p) = Dp \subset Tp\,(M). \end{array}$$

where Dp is an r-dimensional subspace of $Tp(M)$. D is said to be ***differentiable*** if for each point $p \in M$ there exists a neighborhood U of p and r differentiable vector fields on U, say, $X_{(1)}, \cdots\cdots, X_{(r)}$ which form a basis of Dq at every $q \in U$. In this case, $X_1, \cdots\cdots, X_r$ is called a ***local basis*** for the distribution D in U. By a distribution we shall always mean a differentiable distribution.

Example 3. 2. Let X be a vector field of M such that for every point $p \in M$ $Xp \neq 0$. For each point $p \in M$ we put $Dp = \{aXp \mid a \in R\}$. Then, Dp is an 1-dimensional subspace of $Tp(M)$, and thus $D : p \longrightarrow Dp (p \in M)$ is an 1-dimensional distribution. We have to note that the inverse of the above situation is not always true. That is, even if there is an 1-dimensional distribution D there is not necessarily a vector field X such that $Dp=\{aXp \mid a\in R\}$. Suppose the Moebius band $\bar{M}$ as the below figure.



$\bar{M}$.

Since dim $\bar{M}=2$, for each $p\in\bar{M}$ $Tp(\bar{M})$ is a 2-dimensional vector space. That is, if $(U; X^1, X^2)$, U a local coordinate neighborhood of P,

$$Tp(\bar{M})=\{(a\frac{\partial}{\partial x^1},\ b\frac{\partial}{\partial x^2}) \mid a, b\in R\}$$

We put $Dp=\{(o, b\frac{\partial}{\partial x^2}) \mid b\in R\}\subset Tp(\bar{M})$. Then $D: p \rightsquigarrow Dp$ is an 1-dimensional distribution of $\bar{M}$. As the above figure a vector field X of $\bar{M}$ has a point p in $\bar{M}$ such that $Xp=X(p)=0$.

Definition 3.3. Let N be a connected submanifold of M. For an r-dimensional distribution D of M, if $i*(Tp(N))=Dp$ for all $p\in N$ then N is called an integral manifold of the distribution D, where $i: N\rightarrow M$ is the inclusion map. If there is no other integral manifold of D which contains N, then N is called a ***maximal integral manifold*** of D. For each point $p\in M$, if there exists an integral manifold N of D such that $p\in N$, then D is said to be ***integrable.***

Let D be an r-dimensional distribution of M, and let X be a vector field of M. If for each point $p\in M$ $Xp\in Dp$ then X is said to ***belong to D.*** If $[X, Y]$ belongs to D whenever X and Y belong to D, then D is called ***involutive.*** The classical theorem of Frobenius can be formulated such that "A distribution D is integrable $\Longleftrightarrow D$ is invalutive" ([3], [5]).

Proposition 3.4. An r-dimensional distribution is involutive if and only if for each local coordinate neighborhood $(U; x^i)$ of M an arbitrary local basis $\{X_1, \cdots, X_r\}$ of D on U satisfies.

$$[X_\alpha, X_\beta]=\sum_\gamma F^\gamma_{\alpha\beta}X_\gamma, \quad (\alpha, \beta=1, 2, \cdots\cdots, r)$$

where $F^{r}_{\alpha\beta}: U \longrightarrow \mathbf{R}$ is of C^{∞}-class.

Proof. If vector fields X and Y defined on U belong to D, then we have

$$[X, Y] \text{ belongs to } D \Rightarrow [X, Y]=\sum_{i=1}^{r} f^{i}_{X,Y} X_{(i)},$$

where $f^{i}_{X,Y} \in C^{\infty}(U)$ for all $i=1, \cdots, r$. Since $X_{(\alpha)}$ and $X_{(\beta)}$ $(\alpha, \beta=1, \cdots, r)$ belong to D we have

$$[X_{(\alpha)}, X_{(\beta)}]=\sum_{\gamma=1}^{r} F^{r}_{\alpha\beta} X_{(\gamma)}.$$

Conversely, let X and Y defined on U belong to D, then

$$X=\sum_{\gamma=1}^{r} f^{\gamma} X_{(\gamma)},\ Y=\sum_{\gamma=1}^{r} g^{\gamma} X_{(\gamma)},\quad (f^{\gamma}, g^{\gamma} \in C^{\infty}(U)).$$

In this case,

$$[X, Y]=\sum_{\alpha=1}^{r} \sum_{\beta=1}^{r} f^{\alpha} g^{\beta} [X_{(\alpha)}, X_{(\beta)}]$$

$$=\sum_{\alpha=1}^{r} \sum_{\beta=1}^{r} \sum_{\gamma=1}^{r} f^{\alpha} g^{\beta} F^{r}_{\gamma\beta} X_{(\gamma)}$$

which belongs to D. Hence D is involutive. *q. e. d.*

In the process of a proof of the Frobenius theorem above, it is proved that if D is integrable then each integral manifold of D is given by the equation

$$x^{r+1}=C^{r+1}, \cdots,\ x^{n}=C^{n}\ (C^{r+1}, \cdots,\ C^{n}: \text{constants})$$

on each small local coordinate neighborhood $(U; x^{h})$ ([2]).

Definition 3.5. Let D be an n-dimensional distribution on a Riemannian manifold (M, g). For a vector field $X \in *(M)$ and a vector field Y which belongs to D, if $\nabla_X Y$ always belongs to D then D is said to be parallel ([9], [10], [11]).

Proposition 3.6. A parallel distribution D on a Riemannian manifold (M, g) is integrable.

Proof. For vector fields X and Y which belong to D, we have

$$[X, Y]=\nabla_X Y-\nabla_Y X \text{(The upper of Definition 2.7)}$$

Since D is parallel, $\nabla_X Y$ and $\nabla_Y X$ belong to D, and thus $[X, Y]$ belongs to D. This implies that D is involutive. By the Frobenius theorem above. D is integral. *q. e. d.*

By $X \in D$ we shall mean that X belongs to D, where X is a vector field and D a distribution.

Proposition 3.7. Let D be a distribution on a Riemannian manifold If for $X, Y \in D$ $\nabla_X Y \in D$ then D is integrable.

Proof. For $X, Y \in D$

$$[X, Y]=\nabla_X Y-\nabla_Y X.$$

Since $\nabla_X Y$ and $\nabla_Y X$ belon to D by our hypothesis, $[X, Y]$ belongs to D. Hence D is involutive, and by Frobenius theorem D is integrable. *q. e. d.*

Let D be an r-dimensional distribution on a Riemannian manifold (M, g). For each point $p \in M$ we put

$$D_p^{\perp} = \{X \in T_p(M) \mid \text{for all } Y \in D \ \langle X, Y \rangle_p = 0\}.$$

which is called the ***orthogonal complement*** of D. It is clear that $\dim D_p^{\perp} = n - r$. Therefore, the correspondence $p \longrightarrow D_p^{\perp}$ is a distribution $D^{\perp}$.

Proposition 3. 8. Let D be an r-dimensional distribution on an n-dimensional Riemannian manifold (M, g). If D is parallel, then so is $D^{\perp}$.

Proof. For $Y \in D$ and $Z \in D^{\perp}$ we have $\langle Y, Z \rangle = 0$.
Then, for $X \in *(M)$ (Definition 2. 5)

$$\nabla_X(\langle Y, Z \rangle) = \langle \nabla_X Y, Z \rangle + \langle Y, \nabla_X Z \rangle = 0$$

Since $\nabla_X Y \in D$ we have $\langle \nabla_X Y, Z \rangle = 0$, and thus $\langle Y, \nabla_X Z \rangle = 0$. Since Y is an arbitrary vector field which belongs to D, $\nabla_X Z \in D^{\perp}$.
Therefore $D^{\perp}$ is also parallel. *q. e. d.*

Let D be an r-dimensional distribution on an n-dimensional Riemannian manifold (M, g). If D is parallel, by Proposition 3. 6 and Proposition 3. 8, D and $D^{\perp}$ have integral manifolds, respectively. Then, there is a local coordinate system $\{(U ; X^h)\}$ of M such that

(i) the equation of the integral manifold of D is given by

$$X^{r+1} = C^{r+1}, \cdots, X^n = C^n \ (C^{r+1}, \cdots, C^n : \text{constants})$$

(ii) the equation of the integral manifold of $D^{\perp}$ is given by

$$X^1 = b^1, \cdots, X^r = b^r \ (b^1, \cdots, b^r : \text{constants})$$

in U ([9], [10], [11]). Then, a local basis of D on U is the set $\{\frac{\partial}{\partial x^1}, \cdots, \frac{\partial}{\partial x^r}\}$, and a local basis of $D^{\perp}$ on U is the set $\{\frac{\partial}{\partial x^{r+1}}, \cdots, \frac{\partial}{\partial x^n}\}$.

Therefore, for each point $p \in U$

$$g(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j})_p = g_{ij}(p) = 0$$

if $1 \le i \le r$ and $r + 1 \le j \le n$. Therefore, in $(U : X^h)$, the components of the Riemannian metric g is represented by the matrix form :

$$(g_{ji}) = \begin{pmatrix} g_{\beta\alpha} & 0 \\ 0 & g_{\mu\lambda} \end{pmatrix} \quad (1 \le \alpha, \beta \le r, \ \ r + 1 \le \mu, \lambda \le n) \qquad (※)$$

Since D is parallel and $\{\frac{\partial}{\partial x^1}, \cdots\cdots, \frac{\partial}{\partial x^r}\}$ is a local basis of D.

$$\nabla_{e_\beta} e_\alpha = \left\{ {\gamma \atop \beta\alpha} \right\} e_r, \quad e_\alpha = \frac{\partial}{\partial x^\alpha}, \quad \alpha, \beta = 1, 2, \cdots\cdots, r$$

belongs to D. Therefore, we have

$$\left\{ {\gamma \atop \alpha\beta} \right\} = 0 \quad (\gamma = r+1, \cdots, n, \quad \alpha, \beta = 1, 2, \cdots, r). \tag{※※}$$

Similarly, we have the following;

$$\left\{ {\gamma \atop \mu\lambda} \right\} = 0 \quad (\gamma = 1, 2, \cdots, r, \quad \mu, \lambda = r+1, \cdots, n). \tag{※※※}$$

Proposition 3.9. Unter the above circumstance,

(i) $g_{\alpha\beta}$ $(1 \le \alpha, \beta \le r)$ have nothing to do with $X^{r+1}, \cdots,$ and X^n,

(ii) $g_{\mu\lambda}$ $(r+1 \le \mu, \lambda \le n)$ have nothing to do with $X^1, \cdots,$ and X^r.

Proof. Since D and $D^\perp$ are parallel, for e_α $(1 \le \alpha \le r) \in D$ and e_λ $(r+1 \le \lambda \le n) \in D^\perp$, we have the following;

$$\nabla_{e_\lambda} e_\alpha = \left\{ {m \atop \lambda\alpha} \right\} e_m \in D, \quad \nabla_{e_\alpha} e_\lambda = \left\{ {m \atop \alpha\lambda} \right\} e_m \in D^\perp.$$

Noting that ∇ is the Riemannian connection $\left\{ {h \atop ji} \right\} = \left\{ {h \atop ij} \right\}$ and thus we have $\left\{ {m \atop \alpha\lambda} \right\} = \left\{ {m \atop \lambda\alpha} \right\} = 0$

by (※※) and (※※※). Since

$$\frac{\partial g_{\beta\alpha}}{\partial x^\lambda} = \left\{ {m \atop \lambda\beta} \right\} g_{m\alpha} + \left\{ {m \atop \lambda\alpha} \right\} g_{m\beta} \quad \text{(Proposition 2.6)},$$

and, for $\lambda (r+1 \le \lambda \le n)$ and $\alpha, \beta (1 \le \alpha, \beta \le r)$ $\left\{ {m \atop \lambda\beta} \right\} = 0 = \left\{ {m \atop \lambda\alpha} \right\}$.

we have

$$\frac{\partial g_{\beta\alpha}}{\partial x^\lambda} = 0 \quad (1 \le \alpha, \beta \le r, \quad r+1 \le \lambda \le n).$$

This implies that $g_{\beta\alpha}$ $(1 \le \alpha, \beta \le r)$ are independent of $X^{r+1}, \cdots\cdots,$ and X^n. Similarly, we can prove (ii). *q. e. d.*

Proposition 3.10. Let (M, g) be an n-dimensional Riemannian manifold, and D be an r-dimensional distribution of M which is parallel. Then, the line limit ds of M is represented by

$$ds^2 = ds_1^2 + ds_2^2,$$

where ds_1 is a line element of an r-dimensional Riemannian manifold and ds_2 a line element of an $(n-r)$-dimensional Riemannian manifold.

Proof. As the above description, M has a local coordinate system $\{(U : X^h)\}$ such that on U

$$X^{r+1} = C^{r+1}, \cdots, X^n = C^n \; (C^{r+1}, \cdots, C^n : \text{constants})$$

is the equation of the integral manifold of D and

$$X^1 = b^1, \cdots, X^r = b^r \ (b^1, \cdots, b^r : \text{constants})$$

is the equation of the integral manifold of $D^\perp$.

For $p_0 \in U$, let $(x_0^1, \cdots\cdots, x_0^n)$ be the local coordinate of p_0.
For a small positive number ε we put

$N = \{p \in U \mid |x^h - x_0^h| < \varepsilon, h = 1, \cdots, n, (x^1, x^2, \cdots\cdots, x^n)$ is the local coordinate of $p\}$, then N is open in M. We also put

$$P = \{(x^1, \cdots, x^r) \mid |x^h - x_0^h| < \varepsilon, h = 1, \cdots, r. \ (x^1, \cdots, x^r) \in \boldsymbol{R}^r\}$$

$$Q = \{(x^{r+1}, \cdots, x^n) \mid |x^h - x_0^h| < \varepsilon, h = r+1, \cdots, n. \ (x^{r+1}, \cdots, x^n) \in \boldsymbol{R}^{n-r}\},$$

where $(x_0^1, \cdots, x_0^n)$ is a fixed point in $\boldsymbol{R}^n$, then P and Q are open in $\boldsymbol{R}^r$ and $\boldsymbol{R}^{n-r}$, respectively.

Let g_1 be the Riemannian metric on P which has $g_{\alpha\beta}(x^1, \cdots, x^r)$ $(1 \leq \alpha, \beta \leq r)$ as its components, and let g_2 be the Riemannian metric on Q which has $g_{\mu\lambda}(x^{r+1}, \cdots, x^n)$ $(r+1 \leq \mu, \lambda \leq n)$ as its components. Then there is the C^∞-homeomorphism

$$\begin{array}{ccc} \varphi : (N, g) & \xrightarrow{\cong} & (P, g_1) \times (Q, g_2) \\ \Psi & & \Psi \\ P(x^1, \cdots, x^n) & \longmapsto & (x^1, \cdots, x^r) \times (x^{r+1}, \cdots, x^n). \end{array}$$

That is, (M, g) is locally a product of an r-dimensional Riemannian space and an $(n-r)$-dimensional Riemannian space. In this case, since

$$(g_{ji}) = \begin{pmatrix} g_{\alpha\beta}(x^1, \cdots, x^r) & 0 \\ 0 & g_{\mu\lambda}(x^{r+1}, \cdots, x^n) \end{pmatrix}$$

we have the following ;

$$\begin{aligned} ds^2 = g_{ji}\, dx^j dx^i &= \sum_{\alpha, \beta=1}^{r} g_{\alpha\beta} dx^\alpha dx^\beta + \sum_{\mu, \lambda=r+1}^{n} g_{\mu\lambda} dx^\mu dx^\lambda \\ &= ds_1^2 + ds_2^2, \end{aligned}$$

where ds_1 is the line element of P and ds_2 the element of Q. *q. e. d.*

Example 3. 11. Suppose that $(M, g) = (P, g_1) \times (Q, g_2)$, where M, P and Q are Riamannian mamfolds such that dim $M = n$, dim $P = r$ and dim $Q = n - r$.

Take a point $(p, q) \in M = P \times Q$ and define

$$(P, q) = \{(s, q) \mid s \in \underline{P}\}, \quad (p, Q) = \{(p, s) \mid s \in Q\},$$

Then, $(\underline{P}, q)$ $(\cong \underline{p})$ and (p, Q) $(\cong Q)$ are submanifolds of M. If we put $T_{(p,q)}(P, \underline{q}) = D_{(p,q)}$, then $D : s\,(\in p) \longrightarrow D_{(s,q)}$ is an r-dimensional distribution on p. If we also put $T_{(p,q)}(p, Q) = D'_{(p,q)}$, then $D' : s\,(\in Q) \longrightarrow D_{(p,s)}$ is an $(n-r)$- dimensional distribution on Q. In this case

(i) $D' = D$

(ii) D and D' are parallel

Proof. Let $(U : X^h)$ be a coordinate neighborhood of (p, q) in M, and let $(x_0^1, \cdots, x_0^n)$ be the local coordinate of $(p, q) \in M = P \times Q$. Then $(x_0^1, \cdots, x_0^r)$ is the local coordinate of $p \in P$, and $(x_0^{r+1}, \cdots x_0^n)$ is the local coordinate of $q \in Q$. In this case, the Riemannian metric g_1 on P has $g_{\alpha\beta}(x^1, \cdots, x^r)$ $(1 \leq \alpha, \beta \leq r)$ as its components and the Riemannian metric g_2 has $g_{\mu\lambda}(x^{r+1}, \cdots, x^n)$ $(r+1 \leq \mu, \lambda \leq n)$ as its components. Then, the components $\{g_{ji}\}$ of g is represented by

$$(g_{ji}) = \begin{pmatrix} g_{\alpha\beta}(x^1, \cdots, x^r) & 0 \\ 0 & g_{\mu\lambda}(x^{r+1}, \cdots, x^n) \end{pmatrix}$$

This implies that

(a) $\langle \frac{\partial}{\partial x^{\alpha}}, \frac{\partial}{\partial x^{\lambda}} \rangle = (1 \leq \alpha \leq r, \gamma + 1 \leq \lambda \leq n)$

(b) $\frac{\partial g_{\mu\lambda}}{\partial x^{\alpha}} = 0 \quad (1 \leq \alpha \leq r, r+1 \leq \mu, \lambda \leq n)$

It follows that $D^{\perp} = D'$. Since

$$0 = \frac{\partial g_{\mu\lambda}}{\partial x^{\alpha}} = \left\{ {m \atop \alpha\mu} \right\} g_{m\lambda} + \left\{ {m \atop \alpha\lambda} \right\} g_{m\mu} \quad (1 \leq \alpha \leq r, r+1 \leq \mu, \lambda \leq n)$$

and for $m \leq r$ $g_{m\lambda} = 0 = g_{m\mu}$, we have

$$\sum_{m=r+1}^{n} \left(\left\{ {m \atop \alpha\mu} \right\} g_{m\lambda} + \left\{ {m \atop \alpha\lambda} \right\} g_{m\mu} \right) = 0$$

Therefore, if $\lambda = \mu$ then $\left\{ {m \atop \alpha\mu} \right\} = \left\{ {m \atop \mu\alpha} \right\} = 0$ for $r+1 \leq \mu \leq n$, $1 \leq \alpha \leq r$ and $m \geq r+1$

For $1 \leq \alpha \leq r$ and $r+1 \leq \lambda \leq n$

$$\nabla_{\frac{\partial}{\partial x^{\lambda}}} \frac{\partial}{\partial x^{\alpha}} = \left\{ {m \atop \lambda\alpha} \right\} \frac{\partial}{\partial x^{m}} \in D,$$

because of that for $m \geq r+1$ $\left\{ {m \atop \lambda\alpha} \right\} = 0$. Therefore, D is parallel, and thus $D^{\perp} = D'$ is parallel by Proposition 3. 8. *q. e. d.*

§ 4. Holonomy Groups

Let (M, g) be an n-dimensional Riemannian manifold.

Definition 4. 1. For a closed interval I a C^{∞} curve $c : I \longrightarrow M$ is called a *curve*. For a curve c and each point $c(t)$ $(t \in I)$, the correspondence $v : c(t) \rightsquigarrow v(t) \in T_{c(t)}(M)$ is called a ***vector field belonging to the curve*** c, and written v. Therefore, if we take a coordinate neighborhood $(U ; X^h)$ of $c(t) \in M$, we have

$$v(t) = v^i(t) \frac{\partial}{\partial x^i}$$

in U, where $v^i(t)$ is of C^∞ class with respect to t. In this case, if we take $(x^1(t), \cdots\cdots, x^n(t))$ as the local coordinate of $c(t)$ in U, then we have the following;

$$\nabla_c v = \frac{Dv^h}{dt}\left(\frac{\partial}{\partial x^h}\right),$$

where $$\frac{dv^h}{dt} = \frac{dv^h}{dt} + \left\{ {h \atop ji} \right\} \frac{dx^j}{dt} v^i.$$

$\nabla_c v$ is called the ***covariant differential*** of v belonging to the curve c. If $\nabla_c v = 0$ then v is said to be *parallel* belonging to the curve c.

Let us take a vector field $X \in *(M)$. The restriction of X on c will be denoted by $X \mid c$. If we put $v = X \mid c$ then

$$(\nabla_c v)_{c(t)} = \left(\frac{dx^i}{dt} + \left\{ {h \atop ji} \right\} \frac{dx^j}{dt} X^i\right)_{c(t)} \frac{\partial}{\partial x^h}$$

where $X = X^h \dfrac{\partial}{\partial x^h}$ in U. If $\nabla_c v = 0$, *i. e.*,

$$\frac{dX^h}{dt} + \left\{ {h \atop ji} \right\} X^i \frac{dx^j}{dt} = 0,$$

then X is said to be parallel bolonging to the curve c.

Definition 4. 2. For a curve $c : [a, b] \longrightarrow M$ we define

$$c^{-1} : [a, b] \longrightarrow M$$

by $c^{-1}(t) = c(a + b - t)$ for all $t \in [a, b]$. Given two curves

$$c_1 : [a, b] \longrightarrow M, \quad c_2 : [b, f] \longrightarrow M$$

such that $c_1(b) = c_2(b)$, then we define $c_2 c_1 = [a, f) \longrightarrow M$ such that

$$c_2 c_1(t) = \begin{array}{ll} c_1(t), & t \in [a, b] \\ c_2(t), & t \in [b, f], \end{array}$$

which is a continuous function. By the same way as above, for curves $c_1, \cdots\cdots, c_r$ we can define a continuous function $c_r c_{r-1} \cdots\cdots c_2 c_1$ which is called a ***piecewise curve***.

Definition 4. 3. Let $c : [a, b] \longrightarrow M$ be a curve, and let v be a parallel vector field belonging to c. For a local representation $c(t) = (x^1(t), \cdots\cdots, x^n(t))$ v satisfies the following differential equation

$$\frac{dv^h}{dt} + \left\{ {h \atop ji} \right\} \frac{dx^j}{dt} v^i = 0 \qquad (※)$$

Conversely, when a primitive condition is given the equation (※) has only on solution. Therefore, if we take a tangent vector $A \in T_{c(a)}(M)$, then there is only one parallel vector field $v(t)$ such that $v(a) = A$. That is $v(t)$ is the solution of

(※) with $v(a)=A$. We define a linear map.

$$\begin{array}{ccc} \Pi_c : T_{c(a)}(M) & \longrightarrow & T_{c(b)}(M) \\ \Cup & & \Cup \\ v(a) & \longmapsto & v(b) \\ \| & & \| \\ A & \longmapsto & B \end{array}$$

which is called the ***parallel transformation*** belonging to c. The following properties hold ([1], [3], [8]).

1°. Π_c is an isomorphism

2°. For $A, A' \in T_{c(a)}(M)$ $\quad \langle A, A' \rangle_{c(a)} = \langle \Pi_c(A), \Pi_c(A') \rangle_{c(b)}$

3°. $\Pi_{c^{-1}} = (\Pi_c)^{-1}$

4°. $\Pi_{c_2 c_1} = \Pi_{c_2} \circ \Pi_{c_1}$,

where for a coordinate neighborhood $(U : X^h)$ $c_2 c_1 : I \longrightarrow U$. For a piecewise curve $c_r \cdots\cdots c_1$ the same things as above are hold.

Let us consider a Riemannian manifold (M, g) and its the Riemannian connection ∇ (Definition 2.5). For a vector field $X \in *(M)$, if $\nabla X = 0$, then X is called a ***parallel vector field***. By this definition we can easily see the following:

(i) If X is a parallel vector field then X is parallel belonging to every curve c.

(ii) If X is parallel belonging to arbitrary curve then X is a parallel vector field.

Proposition 4.4. Let (M, g) be a Riemannian manifold with dim $M=n$. For a curve $c : I \longrightarrow M$, let us consider two vector fields v and w belonging to the curve c. The following hold;

(i) $\nabla_{\dot c} g = 0$, $\nabla_{\dot c} \langle v, w \rangle = \langle \nabla_{\dot c} v, w \rangle + \langle v, \nabla_{\dot c} w \rangle$

(ii) If v is parallel then $\| v \|$ is constant on the curve c.

(iii) If v and w are parallel then $\langle v, w \rangle$ is constant on the curve c.

Proof. Let $(x^1(t), \cdots\cdots, x^n(t))$ be the local coordinate of $c(t)$.

$$\nabla_{\dot c} g = \nabla_{\dot c} g_{ji}\, dx^j \otimes dx^i - g_{ji} \frac{dx^k}{dt} \left\{ {j \atop kl} \right\} dx^l \otimes dx^i - g_{ji} \frac{dx^k}{dt} dx^j$$

$$\otimes\, dx^l \left\{ {i \atop kl} \right\} \quad \nabla_{\dot c} g_{ji}\, dx^j \otimes dx^i - \frac{dx^k}{dt} \left\{ {m \atop kj} \right\} g_{mi}\, dx^j \otimes dx^i$$

$$- \frac{dx^k}{dt} \left\{ {m \atop ki} \right\} g_{jm}\, dx^j \otimes dx^i$$

$$\nabla_{\dot c} g_{ji}\, dx^j \otimes dx^i = \frac{dx^k}{dt} \frac{\partial g_{ji}}{\partial x^k} dx^j \otimes dx^i$$

$$=\frac{dx^k}{dt}\left[\left\{{m \atop kj}\right\} g_{mi} + \left\{{m \atop ki}\right\} g_{jm}\right] dx^j \otimes dx^i$$

$$\therefore\ \nabla_{\dot c}\, g = 0$$

Let $v = v^h(t)\frac{\partial}{\partial x^h}$ and $w = w^h(t)\frac{\partial}{\partial x^h}$ be local representations of v and w, respectively. Since $\langle v, w\rangle = v^j w^i g_{ji}$

$$\nabla_{\dot c(t)}\langle v, w\rangle = \nabla_{\dot c(t)}(v^j w^i g_{ji})$$

$$= \frac{dx^k}{dt}\left(\frac{\partial v^j}{\partial x^k} w^i g_{ji} + v^j \frac{\partial w^i}{\partial x^k} g_{ji} + v^j w^i \frac{\partial g_{ji}}{\partial x^k}\right)$$

$$= \frac{dx^k}{dt}\left(\frac{\partial v^j}{\partial x^k} w^i g_{ji} + v^j \frac{\partial w^i}{\partial x^k} g_{ji} + v^j w^i \left(\left\{{m \atop ki}\right\} g_{jm} + \left\{{m \atop kj}\right\} g_{mi}\right)\right)$$

$$\langle \nabla_{\dot c(t)} v, w\rangle = \left\langle \frac{dx^k}{dt}\left(\frac{\partial v^j}{\partial x^k}\frac{\partial}{\partial x^j} + v^j \left\{{h \atop kj}\right\}\frac{\partial}{\partial x^h}\right),\ w\right\rangle$$

$$= \frac{dx^k}{dt}\left(\frac{\partial v^j}{\partial x^k} w^i g_{ji} + v^j \left\{{h \atop kj}\right\} w^i g_{hi}\right)$$

$$= \frac{dx^k}{dt}\left(\frac{\partial v^j}{\partial x^k} w^i g_{ji} + v^j w^i \left\{{m \atop ki}\right\} g_{jm}\right)$$

Similarly, we have

$$\langle v, \nabla_{\dot c(t)} w\rangle = \frac{dx^k}{dt}\left(v^j \frac{\partial w^i}{\partial x^k} g_{ji} + v^j w^i \left\{{m \atop ki}\right\} g_{jm}\right)$$

$$\therefore\ \nabla_{\dot c(t)}\langle v, w\rangle = \langle \nabla_{\dot c(t)} v, w\rangle + \langle v, \nabla_{\dot c(t)} w\rangle.$$

Hence we proved (i).

(ii) v is a parallel vector field belonging to c. Thus $\Delta_{\dot c(t)} v = 0$. $\|v\|^2 = \langle v, v\rangle$ and

$$\nabla_{\dot c(t)}\langle v, v\rangle = \langle \nabla_{\dot c(t)} v, v\rangle + \langle v, \nabla_{\dot c(t)} v\rangle$$

$$= 0 \quad (\because\ \nabla_{\dot c(t)} v = 0),$$

which means that $\|v\|$ is constant.

(iii) By (i)

$$\nabla_{\dot c}\langle v, w\rangle = \langle \nabla_{\dot c(t)} v, w\rangle + \langle v, \nabla_{\dot c(t)} w\rangle = 0$$

because of that $\nabla_{\dot c(t)} v = 0 = \nabla_{\dot c(t)} w$. This implies that $\langle v, w\rangle$ is contant. *q. e. d.*

Proposition 4. 5. For a Riemannian manifold (M, g), we take two curves c, c' : $[a, b] \longrightarrow M$ such that $c(a) = c'(a) = p \in M$ and $c(b) = c'(b) = q \in M$. Then the following hold.

(i) If $X \in *(M)$ is a parallel vector field belonging to c, then $\Pi_c(X_p) = X_q$.

(ii) If there are n parallel vector fields $X_{(1)}, \cdots\cdots, X_{(n)}$ on (M, g) such that $X_{(1)p}, \cdots\cdots, X_{(n)p}$ are linearly independent, then $\Pi_c = \Pi_c'$.

Proof. Since X is a parallel vector field belonging to c we have $\nabla_{\dot c} X = 0$. If we put $X \mid c = v$, then $\nabla_{\dot c} v = 0$. Therefore, by Definition 4. 3

$\Pi_c(v(a)) = v(b)$.

Since $v(a) = X_p$ and $v(b) = X_q$, we have $\Pi_c(X_p) = X_q$.

Next, let us suppose that $X_{(1)p}, \cdots\cdots, X_{(n)p}$ are linearly independent. Every parallel vector field v belonging to c has the value at the point a;

$$v(a) = a_1 X_{(1)p} + \cdots\cdots + \alpha_n X_{(n)p} \quad (\alpha_i \in \boldsymbol{R}).$$

Since Π_c is a linear map we have

$$\Pi_c(v(a)) = \alpha_1 X_{(1)q} + \cdots\cdots + \alpha_n X_{(n)q}$$

by (i). Similarly, for a parallel vector field w belonging to c', we have

$$w(a) = \beta_1 X_{(1)p} + \cdots\cdots + \beta_n X_{(n)p} \quad (\beta_i \in \boldsymbol{R})$$

and

$$\Pi_{c'} = \beta_1 X_{(1)q} + \cdots\cdots + \beta_n X_{(n)q}$$

Since Π_c and $\Pi_{c'} : T_p(M) \xrightarrow{\cong} T_q(M)$, we have $\Pi_c = \Pi_{c'}$. *q. e. d.*

Definition 4. 6. If the curvature tansor K of a Riemannian manifold (M, g) is zero, then (M, g) is said to be ***flat*** or ***locally flat***.

The following are well known ([6], [7], [8], [10], [11]) :

(i) (M, g) is flat $\Longleftrightarrow$ for each point $p \in M$ there exists a coordinate neighborhood U of p such that there are n parallel vector fields $X_{(1)}, \cdots\cdots, X_{(n)}$ in U which are linearly independent, where dim $M = n$.

(ii) (M, g) is flat $\Longleftrightarrow$ every Christoffel' s symbol $\left\{ {h \atop ji} \right\}$ is zero. (※)

Form the above fact (*i*), it follows that if (M, g) is flat then for each point $p \in M$ there exists an open neighborhood U of p such that for curves $c, c' : [a, b] \longrightarrow U$ $\Pi_c = \Pi_{c'}$. Conversely, if for each point $p \in M$ there exists an open neighborhood U of p as above, then (M, g) is flat. (※※)

Proof. Let dim $M = n$. Consider a curve $c : [a, b] \longrightarrow U$ such that $c(a) = p$ and $c(b) = q$. Then, the linear isomorphism $\Pi_c = \Pi_q : T_p(M) —— T_q(M)$ has nothing to do with taking c. Let $\{e_1, \cdots\cdots, e_n\}$ be a basis of $T_p(M)$. Put $\Pi_q(e_\alpha) = X_{(\alpha)q}$ ($\alpha = 1, 2, \cdots, n$). Then $e_1 = X_{(1)p}, \cdots$, and $e_p = X_{(n)p}$ are linearly independent. Since $X_{(\alpha)}$ ($\alpha = 1, 2, \cdots, n$) is a parallel vector field in U belonging to arbitrary curve on U, by (*i*) above (M, g) is flat. *q. e. d.*

Definition 4. 7. Let (M, g) be a Riemannian manifold with dim $M = n$. For a fixed point $p \in M$, if a curve $c : I \longrightarrow M$ (I : closed interval) satisfies $c(a) = c(b) = p$, then c is called a ***closed curve with initial point p.*** Let τ_p be the set of all closed curves with initial point p. Then τ_p is a group. For an arbitrary curve $c \in \tau_p$ $\Pi_c : T_p(M) \longrightarrow T_p(M)$ satisfies that for A, B in $T_p(M)$

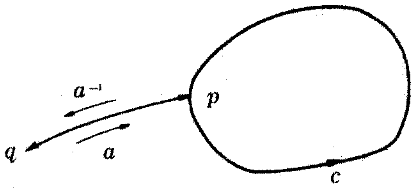
$$\langle A, B\rangle = \langle \Pi_c(A), \ \Pi_c(B)\rangle,$$

that is, Π_c is an orthogonal tronsformation from the metric linear space $T_p(M)$ to itself ((iii) of Proposition 4.4). We put

$$\Phi_p = \{\Pi_c \mid c \in \tau_p\},$$

then Φ_p is the group of orthogonal transformations of $T_p(M)$ Φ_p is called the ***holonomy group*** at p of (M, g). Therefore Φ_p is a subgroup of $O(n)$ which is the orthogonal group of $T_p(M)$.

Let a be a curve on M with initial point q and terimimal point p. Then, for an arbitrary curve $c \in \tau_p$ $a^{-1}ca \in \tau_q$,



Since

$$\Pi_{a^{-1}ca} = \Pi_{a^{-1}} \cdot \Pi_c \cdot \Pi_a$$

we have an isomorphism

$$\begin{array}{ccc} \Phi_p & \longrightarrow & \Phi_q \\ \Psi & & \Psi \\ \Pi_c & \longmapsto & \Pi_{a^{-1}ca} \end{array}$$

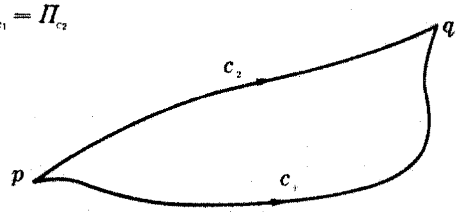
Let a Riemannian manifold (M, g) be connected. For two points p and q in M, there is always a curve c on M such that initial point p and terminal point q. By the above reason

$$\Phi_p \cong \Phi_q.$$

In this case, there is only one holonomy group of (M, g) up to isomorphism. If Φ_p consists of only one element e which is the identity transformation, for $c = c_2^{-1} c_1$ (as the figure below) we have the following ;

$$\Pi_{c_2^{-1}} \cdot \Pi_{c_1} = \Pi_c = e$$

$$\therefore \ \Pi_{c_1} = \Pi_{c_2}$$



Therefore, by (※※) (M, g) is flat. By (※) above, the following is true.

Proposition 4.8. All Christoffel's symbols of (M, g) are zero if and only if each local holonomy groups of (M, g) consists of only elements which is the identity.

§ 5. Main Theorem

Thoughtout this section, by (M, g) we shall mean a Riemannian manifold with dim $M=n$.

Definition 5.1. For each point $q \in M$ and for a vector field $X \in T_q(M)$ there is a geodesic $\gamma : [0, 1] \longrightarrow M$ such that

$$\gamma(0)=q, \qquad \dot{\gamma}(0)=X.$$

If $\gamma(1) \in M$, then we can define a map $\exp_q : T_q(M) \longrightarrow M$ such that $\exp_q(X) = \gamma(1)$, which is called the *exponential map*.

It is well known that for each point $q \in M$ there is a positive number $\varepsilon > 0$ such that

(i) there are an open sphere $B_\varepsilon = \{X \in T_q(M) \mid \|X\| < \varepsilon\}$ of $T_q(M)$ and an open neighborhood U_q of q in M such that

$$exp_q : B_\varepsilon \longrightarrow U_q$$

is a C^∞ homemorphism,

(ii) for every point $p \in U_q$ there exists only one geodesic such that

$$\gamma(0)=q, \qquad \gamma(1)=p.$$

([2], [4], [5]).

Definition 5.2. By (i) above, since

$$\exp_q : B_\varepsilon \longrightarrow U_q$$

is a C^∞ homeomorphism, for each point $p \in U_q$ there is only one vector field $X \in T_q(M)$ such that $\exp_q(X)=p$. If we put $y^h \frac{\partial}{\partial x^h} = X$ there is a correspondence

$$p \longmapsto (y^1, \cdots\cdots, y^n),$$

and thus we have a local coordinate neighborhood $(U_q : Y^h)$ of q. Here, $(U_q : X^h)$ is called a ***normal coordinate system with center*** q. The normal coordinate of q is $(0, \cdots\cdots, 0)$, and $\left\{ {h \atop ji} \right\}_q = 0$ ([3]). That is, a normal coordinate system is a geodesic coordinate system. For $p \in M$ and a small open neighborhood U_p there is always a geodesic coordinate system. *i. e.*, a normal coordinate system.

Proposition 5.3. $(U_p : X^h)$ is a normal coordinate system with center p if and only if $\{{h \atop ij}\} x^j x^i = 0$.

Proof. Since for each geodesic γ in U_p with $\gamma(0)=p$ we can put

$$\gamma : s \longmapsto \exp_p(sX),$$

where $X \in T_p(M)$ with $\| X \| = 1$ and $s < \varepsilon$ we have

$$\gamma(s) = (s\xi^1, s\xi^2, \cdots\cdots\cdots, s\xi^n),$$

where $X = \xi^h \frac{\partial}{\partial x^h}$ (ξ^h $(h=1,2,\cdots,n)$: constants) and $s = \int_0^t \| \dot{r} \| dt < \varepsilon$. Then, $x^h = s\xi^h$ is just the solution of the differential euqation of r :

$$\frac{d^2x^h}{ds^2} + \{{h \atop ji}\} \frac{dx^j}{ds} \frac{dx^i}{ds} = 0 \qquad (※)$$

We see that the resolution of (※) $x^h = s\xi^h$ $(h=1,\cdots,n)$ if and only if

$$\{{h \atop ji}\} (s\xi^1, \cdots, s\xi^n) \xi^j \xi^i = 0 \quad (\because \frac{d^2x^h}{ds^2} = 0)$$

Therefore we have

$$0 = s^2 \{{h \atop ji}\} (s\xi^1, \cdots, s\xi^n) \xi^j \xi^i = \{{h \atop ji}\} (s\xi^j)(s\xi^i)$$

$$= \{{h \atop ji}\} (s\xi^1, \cdots, s\xi^n) x^j x^i. \qquad q.e.d.$$

Let $(U : X^h)$ be a normal coordinate system with center p in (M, g). Then the normal coordinate of p is $(0, \cdots, 0)$ and $\{{h \atop ji}\} = 0$. Take two sets $\{\xi^1, \xi^2, \cdots, \xi^n\}$ and $\{\eta^1, \eta^2, \cdots, \eta^n\}$ of real numbers such that $\xi^h : \eta^h$ $(h=1,2,\cdots,n)$ is not constant. Let ε be a sufficiently small positive number. We define curves c_1, c_2, c_3 and c_4 in U such that

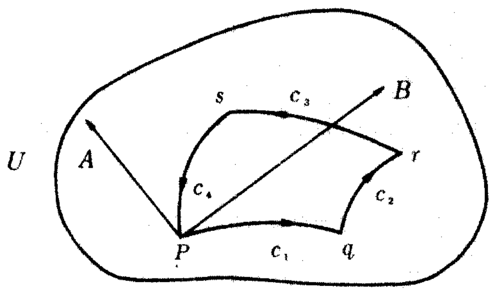
$$c_1(t) = (\varepsilon\xi^1 t, \cdots\cdots\cdots, \varepsilon\xi^n t)$$

$$c_2(t) = (\varepsilon\xi^1 + \varepsilon\eta^1 t, \cdots\cdots\cdots, \varepsilon\xi^n + \varepsilon\eta^n t)$$

$$c_3(t) = (\varepsilon(\xi^1 + \eta^1) - \varepsilon\xi^1 t, \cdots\cdots\cdots, \varepsilon(\xi^n + \eta^n) - (\varepsilon\xi^n t)$$

$$c_4(t) = (\varepsilon\eta^1(1-t), \cdots\cdots\cdots, \varepsilon\eta^n(1-t)),$$

where $0 \leq t \leq 1$ (see the figure below).



In this case, $q=c_1(1)=c_2(0)$ $((\varepsilon\xi^1,\cdots\cdots,\ \varepsilon\xi^n))$, $r=c_2(1)=c_3(0)$ $((\varepsilon(\xi^1+\eta^1)),\cdots,$ $\varepsilon(\xi^n+\eta^n)))$ and $s=c_3(1)=c_4(0)$ $((\varepsilon\eta^1,\cdots\cdots,\ \varepsilon\eta^n))$. Also. $A=\xi^h\frac{\partial}{\partial x^h}$ is the tangent vector to c_1 at p and $B=\eta^h\frac{\partial}{\partial x^h}$ is the tangent vector to c_4 at p. Put $c=c_4 c_3 c_2 c_1$, then $\Pi_c: T_p(M)\longrightarrow T_p(M)$ is represented by the matrix c with

$$c_i^h=\delta_i^h+\varepsilon^2(K_{kji}{}^h)_p\,\xi^k\eta^j+\varepsilon^3(\cdots\cdots\cdots),$$

where $K_{kji}{}^h$ is the components of the curvature tensor K (§2).

This implies that for each $\Pi_c\in\Phi_p$ every component of the matrix form of Π_c is a polynomial of $(K_{kji}{}^h)_p$ ([3], [7]). (※※)

Let $(M,g)=(P,g_1)\times(Q,g_2)$ be an n-dimensional Riemannian manifold, where (P,g_1) is an r-dimensional Riemannian manifold and (Q,g_2) an $(n-r)$-dimensional manifold $(g=g_1+g_2)$. For $(U: x^1,\cdots,x^r)$ and $(V: x^{r+1},\cdots,x^n)$ which are coordinate neighborhoods of P and Q, respectively, $(U\times V: x^1,\cdots,x^r,x^{r+1},\cdots,x^n)$ is a coordinate neighborhood of M.

Let $K_{kji}{}^h$ and $\left\{{h \atop ji}\right\}$ be components of the curvature tensor K and Christoffel's symbols of the Riemannian metric g of M.

Proposition 5.4. Under the above circumstance the following hold,

(i) All Christoffel's symbols except $\left\{{\alpha \atop \gamma\beta}\right\}$ and $\left\{{\lambda \atop \nu\mu}\right\}$ are zero, where $1\leq\alpha,\beta,\gamma\leq r$ and $r+1\leq\lambda,\mu,\nu\leq n$.

(ii) All components except for $K_{\delta\gamma\beta}{}^{\alpha}$ and $K_{\omega\nu\mu}{}^{\lambda}$ are zero, where $1\leq\alpha,\beta,\gamma,\delta\leq r$ and $r+1\leq\lambda,\mu,\nu,\omega\leq n$.

Proof. Since

(a) for $\alpha,\beta=1,\cdots,r$ and $\lambda=r+1,\cdots,n$ $\dfrac{\partial g_{\alpha\beta}}{\partial x^\lambda}=0$

(b) for $\alpha=1,2,\cdots,r$ and $\lambda,\mu=r+1,\cdots,n$ $\dfrac{\partial g_{\lambda\mu}}{\partial x^\alpha}=0$

(c) for $\alpha=1,\cdots,r$ and $\lambda=r+1,\cdots,n$ $g_{\alpha\lambda}=g_{\lambda\alpha}=0$

(d) $\left\{{h \atop ji}\right\}=\frac{1}{2}g^{hk}\left(\dfrac{\partial g_{ik}}{\partial x^j}+\dfrac{\partial g_{jk}}{\partial x^i}-\dfrac{\partial g_{ji}}{\partial x^k}\right)$ (Proposition 2.8)

(e) $K_{kji}{}^h=\dfrac{\partial}{\partial x^k}\left\{{h \atop ji}\right\}-\dfrac{\partial}{\partial x^j}\left\{{h \atop ki}\right\}+\left\{{h \atop km}\right\}\left\{{m \atop ji}\right\}-\left\{{h \atop jm}\right\}\left\{{m \atop ki}\right\}$

our assertion is clear. *q. e. d.*

Theorem 5.5 Let (M,g) be connected. If an r-dimensional space of $T_p(M)$ is invariant by the holonomy group Φ_p at $p\in M$, then there is an r-dimensional parallel distribution D on (M,g). The inverse is true.

Proof. Suppose that an r-dimensional subspace $H^r(P)$ of $T_p(M)$ is invariant by the holonomy group Φ_p at $p\in M$. Take a point $q\in M$ and a curve $c:[a,b]\longrightarrow M$ such that $c(a)=p$ and $c(b)=q$. Then $\Pi_c(H^r(p))=H^r(q)$ is an r- dimensional subspace of $T_q(M)$. We put $D_q=H^r(q)$, then $D: q\rightsquigarrow H^r(q)=D_q$ is an r-dimensional distribution. For $X\in T_p(M)$, $Y\in H^r(p)$ and a closed curve $c:[0,1]\longrightarrow M$ such that $c(0)=c(1)=p$ and $\dot{c}(0)=X$, we consider a parallel vector field v belonging to c with $v(0)=Y$, then

$$0=\nabla_{\dot{c}(0)}\ v(0)=\nabla_X Y\in D_p$$

Therefore D is a parallel distribution with dimension r.

Conversely, let D be an r-dimensional parallel distribution defined on (M, g). Then, as in the proof of Proposition 3. 10, M is locally a product of r-dimensional manifold and $(n-r)$-dimensional Riemannian manifold. Therefore, by Proposition 4. 4 and (※※) above the matrix form C of $\Pi_c\in\Phi_c$ is

$$\begin{pmatrix} A & O \\ O & B \end{pmatrix}$$

where A is an $r\times r$-matrix and B is an $(n-r)\times(n-r)$ matrix. Hence an r-dimensional subspace of $T_p(M)$ is invariant by Π_c. *q. e. d.*

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