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Some Properties of Cohen-Macaulay Rings

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§ 1. Introduction

Recently the properties of Cohen-Macaulay rings are applied to Algebraic Geometry ([5],[6],[7] and [17]). Therefore, a study of Cohen-Macaulay rings takes an important position in Commutative Algebra. We can find basic properties of Cohen-Macaulay Rings *in* [10] and [11].

But Cohen-Macaulay Rings have been studied in several respects. For example,

(i) We can see more significant properties of Cohen-Macaulay rings, in [3],[4],[6],[12],[13] and [14].

(ii) We can see in [8] the relationship between Cohen-Macaulay rings and Hilbert-Samuel polynomials.

(iii) The relationship between Homological dimension and Cohen-Macaulay ring is shown in [9].

The aim of this paper is to define strong-prime divisor (in §3), and to prvoe its existence in Cohen-Macaulay rings (Theorem 4.7), and to prove some properties of Cohen-Macaulay rings.

The detailed contents of this paper are as follows:
In §2, which is a preparation for §4, we describe the definitions of terms which are used in §4, and prove some basic properties of them (Proposition 2.2 to 2.5, Proposition 2.10 and Corollary 2.11).
In §3 we define strong-prime divisor (Definition 3.5) and find some properties of it (Proposition 3.7, 3.8), and in §4 we prove some properties of Cohen-Macaulay rings (Lemma 4.5).
The main theorems of this paper are as follows:

(1) Let $(R, \mathfrak{m})$ be a notherian local ring. Then R is a Cohen-Macaulay

ring if and only if the unmixedness theorem holds in R (Theorem 4.6).

(2) Let $(R, \mathfrak{m})$ be a Cohen-Macaulay ring. If $\mathfrak{a}=(a_1, \cdots, a_r)$ is an ideal of R such that $\mathrm{ht}(\mathfrak{a})=r=\mathrm{ht}(\mathfrak{m})$, then $\mathfrak{m}$ is a strong-prime divisor of $\mathfrak{a}$ (Theorem 4.7).

§ 2. Preliminaries

Let $R=\bigoplus_{n=0}^{\infty} R_n$ be a noetherian graded ring. Then R_0 is a noetherian ring and R is generated by homogeneous elements $x_1, \cdots, x_s$ with degrees $k_1, \cdots, k_s$ (all >0), respectively. For a finitely generated graded R-module M we have a finite number of homogeneous elements $m_1, \cdots, m_t$ with degrees $r_1, \cdots, r_t$, respectively, such that

$$M=Rm_1+\cdots+Rm_t=\bigoplus_{n=0}^{\infty} M_n .$$

In this case, each M_n is finitely generated as an R_0-module, that is, it is generated by all $g_j(x)m_j$ where $g_j(x)$ is a monomial in the x_i of total degree $n-r_j$ ([1]).

Definition 2.1. Let $\mathrm{mod}(R_0)$ be the class of all finitely generated R_0-modules, and let Z be the ring of integers. Then

$$\lambda: \mathrm{mod}(R_0) \rightarrow Z$$

is said to be *additive* if for each short exact sequence

$$0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$$

of finitely generated R_0-modules λ satisties

$$\lambda(M') - \lambda(M) + \lambda(M'') = 0 .$$

For an additive function λ the *Poincaré series* $P(M, t)$ of M is defined by

$$P(M, t)=\sum_{n=0}^{\infty} \lambda(M_n) t^n \in Z[[t]].$$

Proposition 2.2. *We have*

$$P(M, t)=f(t) / \prod_{i=1}^{s}(1-t^{k_i}),$$

where $f(t) \in Z[t]$.

Proof. We will prove by induction on s the number of generators of R over R_0. Let $s=0$, then for all $n>0$, $R_n=0$ and so that $R=R_0$. Hence $M_n=0$ for all sufficiently large n (Note that M is finituly generated R_0-module). Thus $P(M, t)$ is a polynomial.

We assume that $s>0$ and our assertion is true for $s-1$.
It is obvious that the mapping

$$x_s : M_n \longrightarrow M_{n+k_s} \qquad \text{defined by } m \longmapsto x_s m$$

is an R-module homomorphism. If we put $K_n = \mathrm{Ker}\, x_n$, then we have an exact sequence

$$0 \to K_n \to M_n \xrightarrow{x_s} M_{n+k_s} \to L_{n+k_s} \to 0. \quad (*)$$

Put $K=\bigoplus_{n=0}^{\infty} K_n$ and $L=\bigoplus_{n=0}^{\infty} L_n$ then these ar. both finitely generated R-modules since K is a submodule of M and L is a quotient module of M. Furthermore, K and L are annihilated by x_s, and thus they are $R_0[x_1, \cdots, x_{s-1}]$-module. Applying λ to $(*)$ we have

$$\lambda(K_n) - \lambda(M_n) + \lambda(M_{n+k_s}) - \lambda(L_{n+k_s}) = 0$$

(cf. [1]). Therefore

$$\sum_{n=0}^{\infty} t^{n+k_0}\,(\lambda(K_n) - \lambda(M_n) + \lambda(M_{n+k_s}) - \lambda(L_{n+k_s})) = 0$$

that is,

$$t^{k_s}P(K, t) - t^{k_s}P(M, t) + P(M, t) - \sum_{i=0}^{k_s-1}(\lambda(M_i)\,t^i) - P(L, t) + \sum_{i=0}^{k_s-1}\lambda(L_i)t^i = 0.$$

Hence we get the following:

$$(1-t^{k_s})P(M, t) - P(L, t) - t^{k_s}P(K, t) + \sum_{i=0}^{k_s-1}\{\lambda(M_i) - \lambda(L_i)\}t^i = 0.$$

Thus, applying the inductive hypothesis our assertion now follows.

Corollary 2.3. *If each k_i $(1 \le i \le s)$ is equal to 1, then for all sufficiently large n $\lambda(M_n)$ is a polynomial in n (with rational coefficients) of degree $d-1$, where the degree of the zero polynomial is -1.*

Proof. By proposition 2.2. it is clear that

$$\lambda(M_n) = \text{the coefficient of } t^n \text{ in } f(t)(1-t)^{-s}.$$

If there is a term $(1-t)^r$ in $f(t)$ we cancel powers of $(1-t)$ and we may assume $s=d$ and $f(1)\neq 0$. Put $f(t)=\sum_{k=0}^{N} a_k t^k$ where $a_k \in Z$.
Since

$$(1-t)^{-d}=\sum_{k=0}^{\infty} {}_{d+k-1}C_{d-1}\, t^{k}$$

we have

$$\lambda(M_n)=\sum_{k=0}^{N} a_k \cdot {}_{d+n-k-1}C_{d-1} \qquad \text{for all } n \geq N.$$

The right-hand side is a polynomial in n with leading term

$$(\sum a_k)\, n^{d-1} / (d-1)! \neq 0.$$

For a ring R and an R-module M, a chain of submodules of M

$$M=M_0 \supseteq M_1 \supseteq \cdots \supseteq M_n \supseteq \cdots$$

is called a *filtration* of M, and denoted by (M_n). For an ideal $\mathfrak{a}$ of R it is an $\mathfrak{a}$-*filtration* if $\mathfrak{a} M_n \subseteq M_{n+1}$ for all n, and a *stable* $\mathfrak{a}$-*fitration* if $\mathfrak{a} M_n = M_{n+1}$ for all sufficiently large n. Thus $(M_n = \mathfrak{a} M)$ is a stable $\mathfrak{a}$-filtration.

Proposition 2.4. *Let $(R, \mathfrak{m})$ be a noetherian local ring, $\mathfrak{q}$ an $\mathfrak{m}$-primary ideal, M a finitely generated R-module, and (M_n) a stable $\mathfrak{q}$-filtration of M. Then the following hold.*

(i) *$l(M/M_n) < \infty$ for all $n \geq 0$, where $l(M/M_n)$ is the length of (M/M_n).*

(ii) *for all sufficiently large n, the length $l(M/M_n)$ is a polynomial $g(n)$ in n with degree $\leq s$, where s is the least number of generators of $\mathfrak{q}$.*

(iii) *the degree and leading coefficient of $g(n)$ depend only on M and $\mathfrak{q}$, not on the filtration chosen.*

Proof. (i) Put

$$G(R)=\oplus_n \mathfrak{q}^n/\mathfrak{q}^{n+1} \text{ and } G(M)=\oplus_n M_n/M_{n+1}.$$

Then $G_0(R)=R/\mathfrak{q}$ is an artinian local ring ([1]) and each $G_n(M)= M_n/M_{n+1}$ is a noetherian R-module annihilated by $\mathfrak{q}$. Hence $G_n(M)= M_n/M_{n+1}$ is a noetherian $R/\mathfrak{q}$-module with finite length because $R/\mathfrak{q}$ is artinian.

Since

$$l(M/M_n)=\sum_{r=1}^{n} l(M_{r-1}/M_r) \qquad (**)$$

the module M/M_n is of finite length.

(ii) Suppose that $\mathfrak{q}$ is generated by $x_1, \cdots, x_s$, where s is the least number of generators of $\mathfrak{q}$. Then $G(R)$ is generated by $\bar{x}_1, \cdots, \bar{x}_s$, as an $R/\mathfrak{q}$-algebra, where $\bar{x}_i$ is the image of x_i in $\mathfrak{q}/\mathfrak{q}^2$ with degree 1. Since $R/\mathfrak{q}$ is artinian we have

$$\lambda(M_n/M_{n+1}) = l(M_n/M_{n+1})$$

([1]). Hence by Corollary 2.3. $l(M_n/M_{n+1})$ is a polynomial $f(n)$ in n with degree $\leq s-1$ for all large n. By (**)

$$f(n) = l(M_n/M_{n+1}) = l(M/M_{n+1}) - l(M/M_n),$$

and thus $l(M/M_n)$ is a polynomial $g(n)$ with degree $\leq s$ for large n.

(iii) Suppose that $(\widetilde{M}_n)$ is another stable $\mathfrak{q}$-filtration of M. Set

$$\widetilde{g}(n) = l(M/\widetilde{M}_n).$$

Since two-filtrations have bounded difference, thus there exists an integer n_0 such that $M_{n+n_0} \subseteq \widetilde{M}_n$ and $\widetilde{M}_{n+n_0} \subseteq M_n$ for all n ([1]).

we have

$$g(n+n_0) \geq \widetilde{g}(n) \text{ and } \widetilde{g}(n+n_0) \geq g(n).$$

Since $g(n)$ and $\widetilde{g}(n)$ are polynomials in n, consequently we have

$$\lim_{n\to\infty} g(n)/\widetilde{g}(n) = 1$$

for large n.

This implies that g and $\widetilde{g}$ have the same degree and leading coefficient.

Let R be a noetherian semi-local ring, and let

$$\mathfrak{m} = \mathrm{rad}(R).$$

An ideal $\mathfrak{a}$ of R is called an *ideal of definition* of R if its radical is equal to $\mathfrak{m}$.

If $(R, \mathfrak{m})$ is a noetherian local ring, then each ideal of definition of R is primary to $\mathfrak{m}$. In this case, we take $\mathfrak{a}$, an ideal of definition and put

$$G(R) = \oplus_n \mathfrak{q}^n/\mathfrak{q}^{n+1} \text{ and } G^{\mathfrak{a}}(M) = \oplus_n \mathfrak{q}^n M/\mathfrak{q}^{n+1}M.$$

The polynomial $g(n)$ corresponding to the filtration $(\mathfrak{q}^n M)$ is defined by $\chi(M, \mathfrak{q} : n)$, i.e.,

$$\chi(M, \mathfrak{q} : n) = l(M/\mathfrak{q}^n M)$$

for all large n. In particular, we put

$$\chi(R, \mathfrak{q} : n) = \chi(\mathfrak{q} : n).$$

Proposition 2.5. *Let* $(R, \mathfrak{m})$ *be a notherian local ring, and let* $\mathfrak{q}$ *be an* *ideal of definition of R. Then*

$$deg\ \chi(\mathfrak{q}:n) = deg\ \chi(\mathfrak{m}:n).$$

Proof. By our hypothesis there exists an integer r, $r \geq 1$, such that $\mathfrak{m}^r \subseteq \mathfrak{q} \subseteq \mathfrak{m}$. Therefore, we have

$$\mathfrak{m}^{rn} \subseteq \mathfrak{q}^n \subseteq \mathfrak{m}^n$$

and thus the inequalities

$$\chi(\mathfrak{m}:n) \leq \chi(\mathfrak{q}:n) \leq \chi(\mathfrak{m}:rn)$$

hold for all sufficiently large n.

Since the χ's are polynomials in n and

$$\lim_{n\to\infty} \chi(\mathfrak{m}:n) = \lim_{n\to\infty} \chi(\mathfrak{q}:n)$$

we have $deg\ \chi(\mathfrak{m}:n) = deg(\mathfrak{q}:n)$.

Definition 2.6. Let $(R, \mathfrak{m})$ be a noetherian local ring, and let $\mathfrak{a}$ be an ideal of definition of R. For a finitely generated R-module M, we define

$$deg\ \chi(M, \mathfrak{a}:n) = d(M).$$

In purticular,

$$deg\ \chi(\mathfrak{m}:n) = deg\ \chi(\mathfrak{a}:n) = d(R)$$

(See proposition 2.5).

Definition 2.7. Let R be a noetherian ring, and let M be an R-module A prime ideal $\mathfrak{p}$ of R is called an *associated prime* of M if one of the following equivalent conditions holds:

(i) there exists $m \in M$ such that Ann $(m) = \mathfrak{p}$

(ii) there exists an R-module monomorphism $R/\mathfrak{p} \to M$.

In general the set of all associated primes of M is denoted by $\mathrm{Ass}_R(M)$ or Ass (M).

Under the above situation we put

$$\mathrm{Supp}(M) = \{\mathfrak{p} \in \mathrm{Spec}(R) \mid M_{\mathfrak{p}} \neq 0\}$$

then Ass $(M) \subseteq$ Supp (M), and any minimal element of Supp (M) is in Ass (M) ([10]). Conversely, the minimal associated primes of M are the minimal elements of Supp (M). Associated primes which are not minimal are called *embedded primes*.

Definition 2.8. For a ring $R \neq 0$ a finite sequence of $n+1$ prime ideals

$$\mathfrak{p}_0 \supsetneqq \mathfrak{p}_1 \supsetneqq \mathfrak{p}_2 \supsetneqq \cdots \supsetneqq \mathfrak{p}_n$$

in R is called a *prime chain of length n*.

For each prime ideal $\mathfrak{p} \in \operatorname{Spec}(R)$ the supremum of the lengths of the prime chains with $\mathfrak{p}_0 = \mathfrak{p}$ is called the *height* of $\mathfrak{p}$ and it is denoted by $ht(\mathfrak{p})$. Hence $ht(\mathfrak{p}) = 0$ implies that $\mathfrak{p}$ is a minimal prime ideal of R. The dimension of R is defined by

$$\dim(R) = \sup_{\mathfrak{p} \in \operatorname{Spec}(R)} ht(\mathfrak{p})$$

which is called the *Krull dimension* of R.

Let $\mathfrak{a}$ be a proper ideal of R. The height of $\mathfrak{a}$ is defined by

$$ht(\mathfrak{a}) = \inf_{\mathfrak{a} \subset \mathfrak{p} \in \operatorname{Spec}(R)} ht(\mathfrak{p}).$$

It is easy to prove that

(i) for each $\mathfrak{p} \in \operatorname{Spec}(R)$ we have $ht(\mathfrak{p}) = ht(R_{\mathfrak{p}})$

(ii) for each ideal $\mathfrak{a}$ of R

$$\dim(R/\mathfrak{a}) + ht(\mathfrak{a}) \leq \dim(R).$$

Furthermore, for an R-module $M (M \neq 0)$ we define the *dimension* of M by

$$\dim(M) = \dim(R/\operatorname{Ann}(M)).$$

In particular, when $M = 0$ we put $\dim(M) = -1$.

Definition 2.9. Let R be a ring and let M be an R-module. A sequence $a_1, \cdots, a_r$ of elements of R is said to be *M-regular* if for all $i\ (1 \leq i \leq r)$ the element a_i is not a zero divisor of $M/(a_1M + \cdots + a_{i-1}M)$, and

$$M \neq (a_1, \cdots, a_r)M.$$

If all a_i are in ideal $\mathfrak{a}$ of R, $a_1, \cdots, a_r$ is called an *M-regular sequence* in $\mathfrak{a}$. Moreover, if there is no any element $b \in \mathfrak{a}$ such that $a_1, \cdots, a_r, b$ is an M-regular sequence in $\mathfrak{a}$, then $a_1, \cdots, a_r$ is called a *maximal M-regular sequence* in $\mathfrak{a}$. When R is noetherian and M is a finite R-module, and $\mathfrak{a}$ is an ideal of R with $\mathfrak{a}M \neq M$ the length of a maximal M-regular sequence in $\mathfrak{a}$ is called the *$\mathfrak{a}$-depth* of M and is denoted by $depth_{\mathfrak{a}}(M)$. If $(R, \mathfrak{m})$ is a local ring we write $\operatorname{depth}(M)$ (or $\operatorname{depth}_R(M)$) for $\operatorname{depth}_{\mathfrak{m}}(M)$ and call it simply the *depth* of M.

The following are easily proved ([10]).

(i) If $f \in \mathfrak{a}$ is M-regular then $\operatorname{depth}_{\mathfrak{a}}(M/fM) = \operatorname{depth}_R(M) - 1$.

(ii) Let R be a noetherian ring. Then for each finite R-module M and $\mathfrak{p} \in \operatorname{Spec}(R)$

$$\operatorname{depth}(M_{\mathfrak{p}}) = 0 \text{ as } R_{\mathfrak{p}}\text{-module} \Longleftrightarrow \mathfrak{p}R_{\mathfrak{p}} \in \operatorname{Ass}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}) \Longleftrightarrow \mathfrak{p} \in \operatorname{Ass}(M).$$

It consequence

$$\operatorname{depth}(M_{\mathfrak{p}}) \text{ as } R_{\mathfrak{p}}\text{-module} \geq \operatorname{depth}_{\mathfrak{p}}(M).$$

Proposition 2.10. *Let $(R, \mathfrak{m})$ be a noetherian local ring, and let M be a finite R-module such that $M \neq 0$, Then for every $\mathfrak{p} \in \operatorname{Ass}(M)$ we have*

$$depth(M) \leq dim(R/\mathfrak{p}).$$

In particular

$$depth(M) \leq dim(M).$$

Proof. Our proof will be completed by induction on $\dim(R/\mathfrak{p})$. We have to note that

$$\dim(R/\mathfrak{p}) = 0 \Longrightarrow \mathfrak{p} = \mathfrak{m} \Longrightarrow \mathfrak{m} \in \operatorname{Ass}(M) \text{ (by our assumption)} \Longrightarrow \operatorname{depth}(M) = 0.$$

and so $\dim(R/\mathfrak{p}) = 0$ then

$$\operatorname{depth}(M) = 0 = \dim(R/\mathfrak{p}).$$

Next, we assume that $\operatorname{depth}(M) > 0$. Then there exists an M-regular element $f \in \mathfrak{m}$, and $\bigcap f^n M = 0$ because $(R, \mathfrak{m})$ is a noetherian local ring. Therefore, there exists an associated prime ideal $\mathfrak{q}$ of M_1 such that $\mathfrak{q} \supseteq \mathfrak{p} + fR$, where $M_1 = M/fM$ and $\mathfrak{p} \in \operatorname{Ass}(M)$ ([10]). Since f is an M-regular element, $f \notin \mathfrak{p}$, and thus we have $\mathfrak{p} \subsetneq \mathfrak{q}$. This implies that $\dim(R/\mathfrak{p}) > \dim(R/\mathfrak{q})$.

By our induction hypothesis

$$\operatorname{depth}(M) - 1 = \operatorname{depth}(M_1) \leq \dim(R/\mathfrak{q}) < \dim(R/\mathfrak{p}),$$

and thus $\operatorname{depth}(M) \leq \dim(R/\mathfrak{p})$.

Corollary 2.11. *For a noetherian local ring $(R, \mathfrak{m})$ and a finite R-module M, $depth(M) = \infty$ if and only if $M = 0$.*

Proof. Assume that $M \neq 0$.. Then, by proposition 2.10

$$\operatorname{depth}(M) \leq \dim(M).$$

On the other hand, since M is finitely generated, $\dim(M) < \infty$. This implies that $\operatorname{depth}(M) < \infty$. This is a contradiction to our assumption $\operatorname{depth}(M) = \infty$. It follows that $M = 0$.
Conversely, assume that $\operatorname{depth}(M) < \infty$. Then, by proposition 2.10

$$0 \leq \operatorname{depth}(M) \leq \dim(M) < \infty .$$

Since $\dim(0) = -1$ this implies that $M \neq 0$.
Thus the assertion holds.

§ 3. Strong-Prime divisors

In this section, we define maximal strong-prime divisor, strong-prime divisor and find some properties of them.

Lemma 3.1. *Let $\phi : R \to R'$ be a ring homomorphism. If $\mathfrak{q}'$ is primary to $\mathfrak{p}'$ in R' then $\phi^{-1}(\mathfrak{q}') = \mathfrak{q}$ is primary to $\phi^{-1}(\mathfrak{p}') = \mathfrak{p}$.*

Proof. Let x be an element of R. Then we have the following:

$$\begin{aligned} x \text{ is nilpotent module } \mathfrak{q} &\Longleftrightarrow \text{there exists an integer } n \geq 0 \text{ such that } x^n \in \mathfrak{q} \\ &\Longleftrightarrow (\phi(x))^n \in \mathfrak{q}' \\ &\Longleftrightarrow \phi(x) \in \mathfrak{p}' \\ &\Longleftrightarrow x \in \mathfrak{p}. \end{aligned}$$

Hence we see that $\mathfrak{p}$ is the radical of $\mathfrak{q}$. If $ab \in \mathfrak{q}$ and $a \notin \mathfrak{p}$ $(a, b \in R)$, then $\phi(a)\phi(b) \in \mathfrak{q}'$ and $\phi(a) \notin \mathfrak{p}'$.
It follows that $b \in \mathfrak{q}$, and thus we see that $\mathfrak{q}$ is primary to $\mathfrak{p}$.

Let S be a multiplicatively closed subset of a ring R which does not contain zero. We put the following:

$$U = \{ a \in R \mid a \text{ is not } a \text{ zero divisor of } R \}$$

$$\mathfrak{n} = \{ a \in R \mid \text{there exists } s \in S \text{ such that } as = 0 \}.$$

It is easy to prove that $\mathfrak{n}$ is an ideal. Let

$$\phi : R \to R/\mathfrak{n}$$

be the natural homomorphism.

Lemma 3.2. *Under the above situation, let a be an element of the multiplicatively closed set generated by S and U. Then $\phi(a)$ is not a zero divisor of $R/\mathfrak{n}$.*

Proof. We need to note that U is also a multiplicatively closed set. Put

$$a = us, \quad u \in U \text{ and } s \in S.$$

If for an element $b \in R$ $\phi(a)\,\phi(b) = 0$ then $ab = usb \in \mathfrak{n}$. By our hypothesis there exists an element $s' \in S$ such that $usbs' = 0$. Since u is not a zero divisor we have $sbs' = ss'b = 0$. Since $ss' \in S$ we have $b \in \mathfrak{n}$.

It follows that $\phi(b) = 0$. Hence $\phi(a)$ is not a zero divisor of $R/\mathfrak{n}$.

It follows from the definition of $\mathfrak{n}$ that

$$S^{-1}R = \{\, \phi(a)/\phi(s) \mid a \in R,\ s \in S \,\},$$

which is called the *localization* of R with respect to S.

Lemma 3.3. *Let* $\mathfrak{q}$ *be a primary ideal of* R *belonging to a prime ideal*

(1) If $\mathfrak{p} \cap S \neq \phi$, *then* $\mathfrak{p}(S^{-1}R) = \mathfrak{q}(S^{-1}R) = S^{-1}R$.

(2) $\mathfrak{p} \cap S = \phi$. *Then the following hold* : (*a*) $\mathfrak{n} \subset \mathfrak{q}$,

(*b*) $\mathfrak{p}(S^{-1}R)$ *is a prime ideal of* $S^{-1}R$,

(*c*) $\mathfrak{q}(S^{-1}R)$ *is primary to* $\mathfrak{p}(S^{-1}R)$,

(*d*) $\mathfrak{p}(S^{-1}R) \cap R = \mathfrak{p}$ *and* $\mathfrak{q}(S^{-1}R) \cap R = \mathfrak{q}$.

Proof. (1) For $s \in \mathfrak{p} \cap S$ there exists an integer $n\,(n \geq 0)$ such that $s^n \in \mathfrak{q}$.

Since $s^n \in \mathfrak{q}$ we have

$$\mathfrak{q} \cap S = \phi.$$

For each $a/s \in S^{-1}R$ and $s' \in \mathfrak{p} \cap S$ since

$$a/s = as'/ss' \in \mathfrak{p}(S^{-1}R),$$

we have $S^{-1}R = \mathfrak{p}(S^{-1}R)$. Similarly we also have $S^{-1}R = \mathfrak{q}(S^{-1}R)$.

(2) (*a*) Since $\mathfrak{q} \subset \mathfrak{p}$, $\mathfrak{q} \cap S = \phi$. For each $a \in \mathfrak{n}$ there exists an element $s \in S$ such that $as = 0$, hence $0 = as \in \mathfrak{q}$ and thus $a \in \mathfrak{q}$ because $s \notin \mathfrak{q} \subset \mathfrak{p}$. That is, $\mathfrak{n} \subseteq \mathfrak{q}$.

Next we shall prove (*d*). For $b \in \mathfrak{q}(S^{-1}R) \cap R$

$$\bar{\mathfrak{q}}'s/s = b, \quad \bar{\mathfrak{q}}' \in \mathfrak{q} \text{ and } s \in S,$$

in $\mathfrak{q}(S^{-1}R)$. Hence if we put $\bar{\mathfrak{q}}'s = \bar{\mathfrak{q}} \in \mathfrak{q}$ we have

$$\phi(b) = \phi(\bar{\mathfrak{q}})/\phi(s) \quad (\text{Note that } \phi(bs) \in \phi(\mathfrak{q})).$$

Since $\mathfrak{n} \subset \mathfrak{q}$ by (*a*) we have $bs \in \mathfrak{q}$. Since $s \notin \mathfrak{p}$ it follows that $b \in \mathfrak{q}$, and thus $\mathfrak{q}(S^{-1}R) \subset \mathfrak{q}$. Since $\mathfrak{q} \subset \mathfrak{q}(S^{-1}R)$ is obvious,

we have that $\mathfrak{q} = \mathfrak{q}(S^{-1}R) \cap R$. As a particular case, where $\mathfrak{p} = \mathfrak{q}$, we have $\mathfrak{p}=\mathfrak{p}(S^{-1}R)\cap R$ (Note that if $\mathfrak{p}$ is prime then $\mathfrak{p}$ is primary to $\mathfrak{p}$).

$(b),(c)$. Take $\phi(ab)/\phi(st)\in\mathfrak{q}(S^{-1}R)$ such that

$$\phi(a)/\phi(s)\notin\mathfrak{q}(S^{-1}R).$$

Then $ab\in\mathfrak{q}(S^{-1}R)\cap R=\mathfrak{q}$ (by (d)) and $a\notin\mathfrak{q}$. Hence there exists an integer $r(\geq 0)$ such that

$$b^r\in\mathfrak{q},\quad (\phi(b)/\phi(t))^r\in\mathfrak{q}(S^{-1}R).$$

Therefore $\mathfrak{q}(S^{-1}R)$ is a primary ideal. Now, applying this to the case $\mathfrak{q}=\mathfrak{p}$, we see that $\mathfrak{p}(S^{-1}R)$ is a prime ideal, since in this case r can be taken to be 1. This proves (b).

Since elements of $\mathfrak{p}$ is nilpotent modulo $\mathfrak{q}$, elements of $\mathfrak{p}(S^{-1}R)$ are nilpotent modulo $\mathfrak{q}(S^{-1}R)$. That is, $\mathfrak{q}(S^{-1}R)$ belongs to $\mathfrak{p}(S^{-1}R)$. Therefore, (c) holds.

Definition 3.4. For a given ideal $\mathfrak{a}$ of a ring R let

$$\phi: R \rightarrow R/\mathfrak{a}$$

be the natural homomorphism. Let

$$U = \{a\in R \mid \phi(a) \text{ is not a zero divisor of } R/\mathfrak{a}\}.$$

If there exists a prime ideal $\mathfrak{p}$ such that

$$\mathfrak{p}=R-U$$

then $\mathfrak{p}$ is called the *maximal strong-prime divisor* of $\mathfrak{a}$.

If $\mathfrak{p}$ is the maximal strong-prime divisor of $\mathfrak{a}$ then it is obvious that $\mathfrak{p}$ is a maximal prime divisor of $\mathfrak{a}$ ([11]). The converse is not true.

Definition 3.5. Under the situation of Definition 3.4, a prime ideal $\mathfrak{q}$ of R is called a *strong-prime divisor* of $\mathfrak{a}$ if there exists a prime ideal $\mathfrak{p}$ such that $\mathfrak{a}\cap(R-\mathfrak{p})=\phi$ and $\mathfrak{q}R_{\mathfrak{p}}$ is the maximal strong-prime divisor of $\mathfrak{a}R_{\mathfrak{p}}$. If $\mathfrak{q}R_{\mathfrak{p}}$ is a maximal prime divisor of $\mathfrak{a}R_{\mathfrak{p}}$ we say that $\mathfrak{q}$ is a *weakly strong-prime divisor* of $\mathfrak{a}$.

By definition a strong-prime divisor is a weakly strong-prime divisor, and a weakly strong-prime divisor is a prime divisor ([11]). The converses of these statements are not true.

Proposition 3.6. *Any weakly strong-prime divisor of* $\mathfrak{a}$ *contains* $\mathfrak{a}$, *and* *all elements of* $\mathfrak{q}$ *are zero divisors modulo* $\mathfrak{a}$.

Proof. Since $\mathfrak{q}R_\mathfrak{p}$ is a maximal prime divisor of $\mathfrak{a}R_\mathfrak{p}$, by definition of maximal prime divisor $\mathfrak{a}R_\mathfrak{p} \subset \mathfrak{q}R_\mathfrak{p}$ ([11]). Since $\mathfrak{q}R_\mathfrak{p}$ is a prime ideal of $R_\mathfrak{p}$ by (2) of Lemma 3.3. we have

$$\mathfrak{q}R_\mathfrak{p} \cap R = \mathfrak{q}$$

and thus $\mathfrak{a} \subseteq \mathfrak{q}$. Note that $\mathfrak{a} \subset \mathfrak{a}R_\mathfrak{p} \cap R$. As before consider the natural homomorphism

$$\phi : R \to R/\mathfrak{a},$$

and put

$$U = \{ a \in R \mid \phi(a) \text{ is not a zero divisor of } R/\mathfrak{a} \}.$$

Then we can apply Lemma 3.2. to $R/\mathfrak{a}$, and thus we see that each elements of $\mathfrak{q}$ are zero divisors modulo $\mathfrak{a}$.

Proposition 3.7. *A strong-prime divisor* $\mathfrak{p}$ *of* $\mathfrak{a}$ *is the maximal strong-prime divisor of* $\mathfrak{a}$ *if and only if* $\mathfrak{p}$ *is a maximal member of the set of strong-prime divisors of* $\mathfrak{a}$.

Proof. Let $\mathfrak{p}$ be the maximal strong-prime divisor of $\mathfrak{a}$.
Then $U = R - \mathfrak{p}$ is the set of elements of R which are not zero divisor modulo $\mathfrak{a}$. Therefore $\mathfrak{p}R_\mathfrak{p}$ is the maximal strong-prime divisor of $\mathfrak{a}R_\mathfrak{p}$. That is, $\mathfrak{p}$ is a strong-prime divisor of R.
Conversely, let $\mathfrak{q}$ be a maximal member of the set of strong-prime divisors of $\mathfrak{a}$, since a strong-prime divisor is a weakly strong-prime divisor, by proposition 3.6 $\mathfrak{q}$ consists of zero divisor modulo $\mathfrak{a}$. Hence $\mathfrak{q}$ contains in the maximal strong-prime divisor of $\mathfrak{a}$.

Proposition 3.8. *Let* $\mathfrak{p}$ *be the maximal strong-prime divisor of* $\mathfrak{a}$. *Then* $\mathfrak{a}R_\mathfrak{p} \cap R$ *is primary to* $\mathfrak{p}$.

Proof. At first, we have to note that

(i) if $\mathfrak{a}'$ is primary to $\mathfrak{p}'$ then $\mathfrak{p}'$ is the intersection of all prime ideals containing $\mathfrak{a}'$, and thus if $\mathfrak{a}'$ is primary then $\mathfrak{p}'$ is a prime ideal among prime ideals containing $\mathfrak{a}'$.

(ii) for the radical $\mathfrak{p}'$ of $\mathfrak{a}'$, the ideal $\mathfrak{a}'$ is primary ideal if and only if

$ab \in \mathfrak{a}'$ and $b \notin \mathfrak{p}'(a, b \in R)$ implies $a \in \mathfrak{a}'$.

Since $\mathfrak{p}$ is maximal among prime ideals containing $\mathfrak{a}$, the same thing is true for $\mathfrak{p}R_\mathfrak{p}$ and $\mathfrak{a}R_\mathfrak{p}$. Since $R_\mathfrak{p}$ is a local ring with $\mathfrak{p}R_\mathfrak{p}$ as its maximal ideal $\mathfrak{p}R_\mathfrak{p}$ is the radical of $\mathfrak{a}R_\mathfrak{p}$ (see the above statement (i)). By the above statements $\mathfrak{a}R_\mathfrak{p}$ is a primary ideal of $R_\mathfrak{p}$. Therefore $\mathfrak{a}R_\mathfrak{p}$ is primary to $\mathfrak{p}R_\mathfrak{p}$.

By Lemma 3.1, $\mathfrak{a}R_\mathfrak{p} \cap R$ is primary to $\mathfrak{p} = \mathfrak{p}R_\mathfrak{p} \cap R$ (see (2) of Lemma 3.3).

§4. Cohen-Macaulay Rings

Throughout this section, by $(R, \mathfrak{m})$ we mean a noetherian local ring without any statements. As in Proposition 2.10 for a finite R-module M $(\neq 0)$ we have depth$(M) \leq$ dim(M).

Definition 4.1 A finite R-module M is called a *Cohen-Macaulay* R-module if $M=0$ or depth(M) = dim(M). If the local ring $(R, \mathfrak{m})$ is a Cohen-Macaulay R-module then we say that R is a Cohen-Macaulay ring.

Proposition 4.2 *For a Cohen-Macanlay ring* $(R, \mathfrak{m})$, *we have*

$$d(R) \geq dim(R) = depth(R).$$

Proof. We will prove by induction on $d(R)$ (for notation $d(R)$ see §2). Assume that $d(R) = 0$. In the graded ring

$$G(R) = \oplus_n \mathfrak{m}^n/\mathfrak{m}^{n+1},$$

deg $\chi(\mathfrak{m} : n) = 0$ for large n. This implies that large $\nu, \mathfrak{m}^\nu = \mathfrak{m}^{\nu+1} = \cdots$, and thus $\mathfrak{m}^\nu = (0)$ ([10]). Therefore, it follows that the length $l(R)$ of R is finite, and thus R is artinian. Therefore, we have dim$(R) = 0$.

Next, suppose $d(R) > 0$. If dim$(R) = 0$ our assertion is obvious. Hence we assume that dim$(R) > 0$. Then there exists a prime chain of length $e > 0$ such that

$$\mathfrak{p}_0 \supsetneqq \mathfrak{p}_1 \supsetneqq \cdots \supsetneqq \mathfrak{p}_e = \mathfrak{p}.$$

If we take an element $x \in \mathfrak{p}_{e-1} - \mathfrak{p}_e = \mathfrak{p}$, then dim$(R/(xR + \mathfrak{p})) \geq e - 1$. Suppose the exact sequence

$$0 \to R/\mathfrak{p} \xrightarrow{x} R/\mathfrak{p} \to R/(xR + \mathfrak{p}) \to 0.$$

Since $\chi(R/(xR + \mathfrak{p}), \mathfrak{m} : n)$ is a polynomial of degree $< d(R/\mathfrak{p})$(cf. [10]) we have the following:

$$d(R/(xR+\mathfrak{p}) < d(R/\mathfrak{p}) \leq d(R).$$

Therefore, by induction hypothesis, we have

$$e-1 \leq \dim(R/(xR+\mathfrak{p}) \leq d(R/(xR+\mathfrak{p}) \leq d(R),$$

and thus $\dim(R) \leq d(R)$.

Lemma 4.3 *Let R be a noetherian ring. If M is a finite R-module the $Ass(M)$ is a finite set.*

Proof. Under our situation there exists a chain of submodules

$$(0) = M_0 \subsetneqq \cdots \subsetneqq M_{n-1} \subsetneqq M_n = M$$

such that $M_i/M_{i-1} \cong R/\mathfrak{p}_i$ for some $\mathfrak{p}_i \in \mathrm{spec}(R)$ for $1 \leq i \leq n$ ([10]).
Since for an exact sequence $0 \to M' \to M'' \to M'''$ of R-modules

$$\mathrm{Ass}(M'') \subseteq \mathrm{Ass}(M') \cup \mathrm{Ass}(M''')$$

we have

$$\mathrm{Ass}(M) \subseteq \mathrm{Ass}(M_1) \cup \mathrm{Ass}(M_2/M_1) \cup \cdots \cup \mathrm{Ass}(M/M_{n-1}).$$

In particular,

$$\mathrm{Ass}(M_i/M_{i-1}) = \mathrm{Ass}(R/\mathfrak{p}_i) = \{\mathfrak{p}_i\},$$

and so we have

$$\mathrm{Ass}(M) \subseteq \{\mathfrak{p}_1, \cdots\cdots, \mathfrak{p}_n\}.$$

Furthermore using the fact that $\mathrm{Ass}(M) \subseteq \mathrm{Supp}(M)$ and any minimal element of $\mathrm{Supp}(M)$ is in $\mathrm{Ass}(M)$ we can prove that the minimal associated primses of R-module $R/\mathfrak{a}$ are precisely the minimal prime over-ideals of $\mathfrak{a}$ where R is a noetherian ring, $\mathfrak{a}$ an ideal of R and M a finite R-module. In particular, it is well-known that
if $\mathfrak{a} = (a_1, \cdots\cdots, a_r)$ and $\mathfrak{p}$ is a minimal prime over-ideal of $\mathfrak{a}$ then $\mathrm{ht}(\mathfrak{p}) \leq r$ and $\mathrm{ht}(\mathfrak{a}) \leq r$.
Therefore, with together Lemma 4.3 we see that the following definition makes sense.

Definition 4.4. Let R be a noetherian ring and $\mathfrak{a}$ an ideal, and assume that

$$\mathrm{Ass}_R(R/\mathfrak{a}) = \{\mathfrak{p}_1, \cdots\cdots, \mathfrak{p}_s\}.$$

If $\mathrm{ht}(\mathfrak{p}_i) = \mathrm{ht}(\mathfrak{a})$ for all $i = 1, \cdots, n$ then $\mathfrak{a}$ is said to be *unmixed.* If for all r, $r \geq 0$, each ideal $\mathfrak{a} = (a_1, \cdots, a_r)$ with $\mathrm{ht}(\mathfrak{a}) = r$ is unmixed then we say that the *unmixedness theorem holds* in R.

Note that $\mathfrak{a}$ is unmixed$\Leftrightarrow R/\mathfrak{a}$ has no embedded primes.

Lemma 4.5 (i) *If the unmixedness theorem holds in R, then R has no embedded primes.*

(ii) *For each maximal ideal* $\mathfrak{m}$, *we have*

the unmixedness theorem holds in $R \Leftrightarrow$ *the unmixedness theorem holds in* $R_{\mathfrak{m}}$.

Proof. (i) It is obvious that for any ideal $\mathfrak{a}$ of R

$$\dim(R/\mathfrak{a}) + \mathrm{ht}(\mathfrak{a}) \leq \dim(R).$$

Consider the zero ideal (0) Then from

$$\dim(R) + \mathrm{ht}(0) \leq \dim(R)$$

we get $\mathrm{ht}(0)=0$. Hence (0) is unmixed. Since the unmixedness theorem holds in R, all associated primes of $R/(0) = R$ are not embedded primes.

(ii) Since R is noetherian, for each ideal $\mathfrak{a}$ there exists an irredundant primary decomposition

$$\mathfrak{a} = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_s.$$

Then, for a maximal ideal $\mathfrak{m}$ containing $\mathfrak{q}_1, \cdots \mathfrak{q}_s$

$$\mathfrak{a}R_{\mathfrak{m}} = \mathfrak{q}_1 R_{\mathfrak{m}} \cap \cdots \cap \mathfrak{q}_s R_{\mathfrak{m}}$$

is also an irredundant primary decomposition of $\mathfrak{a}R_{\mathfrak{m}}$. Let $\mathfrak{q}_i$ be primary to $\mathfrak{p}_i$ for all $i=1,\cdots s$. Then $\mathfrak{q}_i R_{\mathfrak{m}}$ is also primary to $\mathfrak{p}_i R_{\mathfrak{m}}$. In this case we have

$$\mathrm{Ass}_R(R/\mathfrak{a}) = \{\mathfrak{p}_1, \cdots, \mathfrak{p}_s\}$$

$$\mathrm{Ass}R_{\mathfrak{m}}(R_{\mathfrak{m}}/\mathfrak{a}R_{\mathfrak{m}}) = \{\mathfrak{p}_1 R_{\mathfrak{m}}, \cdots, \mathfrak{p}_s R_{\mathfrak{m}}\} \ ([10]).$$

Moreover, since $1 \in R$ we have the following;

$$\mathfrak{a} = (a_1, \cdots, a_r) \text{ in } R \Leftrightarrow \mathfrak{a}R_{\mathfrak{m}} = (a_1/1, \cdots, a_r/1) \text{ in } R_{\mathfrak{m}},$$

$$\mathrm{ht}(\mathfrak{a}) = r \text{ in } R \Leftrightarrow \mathrm{ht}(\mathfrak{a}R_{\mathfrak{m}}) = r \text{ in } R_{\mathfrak{m}},$$

$\mathfrak{p}_i$ is not an embedded prime$\Leftrightarrow \mathfrak{p}_i R_{\mathfrak{m}}$ is not an embedded prime.

Hence the assertion helds.

Let $(R, \mathfrak{m})$ be a Cohen-Macaulay ring. The following Properties of the local ring $(R, \mathfrak{m})$ have been proved ([10],[11]).

1° For every $\mathfrak{p} \in \mathrm{Ass}(R)$

$$\dim(R/\mathfrak{p}) = \mathrm{depth}(R),$$

Thus R has no embedded Primes.

2° If $a_1, \cdots, a_r$ is an R-regular sequence in $\mathfrak{m}$ then

$$R_r = R/(a_1 R + \cdots + a_r R)$$

is a Cohen-Macaulay ring and $\dim(R_r) = \dim R - r$.

3° For each proper ideal $\mathfrak{a}$ of R

$$\mathrm{ht}(\mathfrak{a}) + \dim(R/\mathfrak{a}) = \dim(R).$$

4° For a sequence $a_1, \cdots, a_r$ of elements in $\mathfrak{m}$ the following conditions are equivalent:

(1) the sequence $a_1, \cdots, a_r$ is R-regular,

(2) $\mathrm{ht}(a_1, \cdots, a_i) = i$ for all $1 \leq i \leq r$,

(3) there exists $a_{r+1}, \cdots, a_n$ ($\dim R = n$) in $\mathfrak{m}$ such that $\{a_1, \cdots, a_n\}$ is a system of parameters, i.e. $(a_1, \cdots, a_n)$ is an ideal of definition,

(4) $\mathrm{ht}(a_1, \cdots, a_r) = r$.

Theorem 4.6 *Let $(R, \mathfrak{m})$ be a noetherian local ring. Then R is a Cohen-Macaulay ring if and only if the unmixedness theorem holds in R.*

Proof. Assume that R is a Cohen-Macaulay ring, at first we will prove that for each $\mathfrak{p} \in \mathrm{Spec}(R)$ $R_\mathfrak{p}$ is a Cohen-Macaulay ring. Assume $\mathfrak{p} \not\supset \mathrm{Ann}(R)$ then $R_\mathfrak{p} = 0$ and thus $R_\mathfrak{p}$ is a Cohen-Macaulay ring. Next we assume that $\mathrm{Ann}(R) \subseteq \mathfrak{p}$ and want to prove that $\dim(R_\mathfrak{p}) = \mathrm{depth}(R_\mathfrak{p})$. Since

$$\dim(R_\mathfrak{p}) = \mathrm{ht}(\mathfrak{p}/\mathrm{Ann}(R)),\ \mathrm{depth} R_\mathfrak{p} = \mathrm{depth}_\mathfrak{p} R.$$

Our proof will be proceeded by induction on $\mathrm{depth}_\mathfrak{p} R$.

(i) $\mathrm{depth}_\mathfrak{p} R = 0$: If $\mathrm{depth}_\mathfrak{p} R = 0$, then $\mathfrak{p}$ is contained in some $\mathfrak{q} \in \mathrm{Ass}(R)$.

By 1°, every associated prime of R is a minimal prime over-ideal of Ann(R). Since $\mathfrak{p} \subseteq \mathfrak{q}$ are in $\mathrm{Ass}(R)$ we have $\mathfrak{p} = \mathfrak{q}$, and thus $\dim R_\mathfrak{p} = \mathrm{ht}(\mathfrak{p}/\mathrm{Ann}(R)) = 0$.

(ii) $\mathrm{depth}_\mathfrak{p} R = r > 0$: We assume that for a prime ideal $\mathfrak{q}$ $\mathrm{depth}_\mathfrak{q} R < r$ then $\dim R_\mathfrak{q} = \mathrm{depth} R_\mathfrak{q}$. Take an R-regular element $a \in \mathfrak{p}$ and put $R_1 = R/aR$. From the exact sequence

$$0 \to R \xrightarrow{a} R$$

we also get the exact sequence

$$0 \to R_\mathfrak{p} \xrightarrow{a} R_\mathfrak{p}.$$

Hence a is a $R_\mathfrak{p}$-regular element. Thus, we have

$$\dim(R_1)_\mathfrak{p} = \dim(R_\mathfrak{p}/aR_\mathfrak{p}) = \dim(R_\mathfrak{p}) - 1$$

and also

$\operatorname{depth}(R_1)_{\mathfrak{p}} = \operatorname{depth} R_{\mathfrak{p}} - 1$.

By 2°, R_1 is a Cohen-Macaulay ring. Hence by our induction hypothesis $\operatorname{depth}(R_1)_{\mathfrak{p}} = \dim(R_1)_{\mathfrak{p}}$. It follows that $\dim(R_{\mathfrak{p}}) = \operatorname{depth}(R_{\mathfrak{p}})$.

Therefore, $R_{\mathfrak{m}}$ is a Cohen-Macaulay local ring for every maximal ideal $\mathfrak{m}$. By using this, we will prove that the unmixedness theorem holds in R. By (ii) of Lemma 4.5 it is sufficient to prove that the unmixedness theorem holds in $R_{\mathfrak{m}}$.

Consider the zero ideal (0) in $R_{\mathfrak{m}}$. Since $R_{\mathfrak{m}}$ is a Cohen-Macaulay ring, the ideal (0) is unmixed. Assume $\operatorname{ht}(a_1, \cdots, a_r) = r > 0$ in $R_{\mathfrak{m}}$.

Then, by 4°, $a_1, \cdots, a_r$ is an $R_{\mathfrak{m}}$-regular sequence. Note that each a_i is in $\mathfrak{m}R_{\mathfrak{m}}$. Hence, by 2°, $R_{\mathfrak{m}}/(a_1R_{\mathfrak{m}} + \cdots + a_rR_{\mathfrak{m}})$ is also a Cohen-Macaulay ring. Also, by 1°, each element $\mathfrak{p}_i$ of $\operatorname{Ass}R_{\mathfrak{m}}(R_{\mathfrak{m}}/a_1R_{\mathfrak{m}} + \cdots + a_rR_{\mathfrak{m}})$ is not an embedded prime. That is, $\operatorname{ht}(\mathfrak{p}_i) = r$.

Conversely assume that unmixedness theorem holds in $(R, \mathfrak{m})$, and let $\operatorname{ht}(\mathfrak{m}) = \dim(R) = r$. By the our assumption we can find an R-regular sequence $a_1, \cdots, a_r$ of elements in $\mathfrak{m}$ such that $\operatorname{ht}(a_1, \cdots, a_i) = i$ and $a_1, \cdots, a_r$ is maximal. Therefore, we have

$$\operatorname{depth}(R) = r = \operatorname{ht}(\mathfrak{m}) = \dim(R).$$

Hence $(R, \mathfrak{m})$ is a Cohen-Macaulay ring.

Theorem 4.7 *Let $(R,\mathfrak{m})$ be a Cohen-Macaulay ring. If $\mathfrak{a} = (a_1, \cdots, a_r)$ is an ideal of R such that $ht(\mathfrak{a}) = r = ht(\mathfrak{m})$, then $\mathfrak{m}$ is the maximal strong-prime divisor of $\mathfrak{a}$ and it is a strong-prime divisor of $\mathfrak{a}$.*

Proof. We have to note that for an ideal $\mathfrak{a}'$ of R the set of all zero divisors of $R/\mathfrak{a}'$ is the set $\bigcup_{\mathfrak{p} \in \operatorname{Ass}(R/\mathfrak{a}')} \mathfrak{p}(\{0\})$. Since R is a Cohen-Macaulay ring, $R_{\mathfrak{m}}$ is also a Cohen-Macaulay local ring.

It is obvious that

$$\operatorname{Ass}_R(R/\mathfrak{a}) = \{\mathfrak{m}\},$$

and thus the set of zero divisor of $R/\mathfrak{a}$ is the set $\mathfrak{m}$.

Therefore, $\mathfrak{m}$ is the maximal strong-prime divisor of $\mathfrak{a}$. By Proposition 3.7, $\mathfrak{m}$ is a strong-prime divisor of $\mathfrak{a}$. (In fact, since $(a_1/1, \cdots, a_r/1) = \mathfrak{a}R_{\mathfrak{m}}$, $\mathfrak{m}R_{\mathfrak{m}}$ is the maximal strong-prime divisor of $\mathfrak{a}R_{\mathfrak{m}}$. Hence $\mathfrak{m}$ is also a strong-prime divisor of $\mathfrak{a}$ (see Definition 3.5)).

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