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Divisors and Holomorphic Line Bundles

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§ 1. Introduction

The theory of functions of several complex variables is closely related to the theory of complex manifolds and algebraic geometry, in particular, to sheaf theory ([1], [3], [4], [5], [8], [12]). Accordingly, investigations on germs of analytic or meromorphic functions defined on complex manifolds are important to study some properties of functions of several complex variables ([2], [7], [9], [13]). In this connection, we need to consider holomorphic line bundles on complex manifolds and divisors which are, as is well known, deeply concerned with holomorphic line bundles.

The purpose of this paper is to make contravariant functors from divisors, holomorphic line bundles and sheaves, and to study some relations between these contravariant functors.

Details of our paper are as follows. In § 2, we make contravariant functors $\mathfrak{M}^*$ and $\mathcal{D}$ (proposition 2.3), and prove that there exists a natural transformation

div : $\mathfrak{M}^* \longrightarrow \mathcal{D}$ (Theorem 2.4)

In § 3, we make a contravariant functor $\mathcal{HLB}$, and prove

i) $\mathfrak{M}^*_c \longrightarrow \mathcal{D}_c \longrightarrow \mathcal{HLB}$ is exact (Theorem 3.7.), and

ii) under some conditions

$$\mathfrak{M}^*_c \longrightarrow \mathcal{D}_c \longrightarrow \mathcal{HLB} \longrightarrow 1$$

is exact (Corollary 3.8 and Corollary 3.9).

In § 4, we make a contravariant functor $\mathcal{SH}$ from sheaf cohomology groups, and prove that there exists a natural isomorphism

$\psi : \mathcal{SH} \longrightarrow \mathcal{HLB}$ (Theorem 4.4).

Throughout this paper C will denote the field of all complex numbers.

§ 2. Divisors

Let Ω be an open subset of C. A function $m: \Omega \rightarrow C$ is a meromorphic function if there exists a discrete subset X of Ω satisfying the following conditions:

i) m is an analytic function on $\Omega \setminus X$, and

ii) every point $x \in X$ is a pole of m.

Let $O(C)$ be the category consisting of all connected open subsets of C and all non-zero meromorphic functions between open subsets of C. Let A_b be the category of all abelian groups and all group homomorphisms. We define a contravariant functor as follows.

$$\mathfrak{M}^* : O(C) \longrightarrow A_b$$

For each $\Omega \in \mathrm{Obj}(O(C))$

$\mathfrak{M}^*(\Omega)$ = the set of all non-zero meromorphic functions defined on Ω,

where $\mathrm{Obj}(O(C))$ is the class of all objects of $O(C)$.

For an open covering $\mathscr{U} = \{U_i \mid i \in I\}$ of Ω, $\{(f_i, g_i) \mid f_i$ and g_i are analytic functions defined on U_i, and $i \in I\}$ defines a meromorphic function m on Ω provided for each point $x \in U_i \cap U_j$

$$f_i / g_i = f_j / g_j$$

where both g_i and g_j are nonzero. In this case, $m \in \mathfrak{M}^*(\Omega)$ if and only if there exist non-zero analytic function $\{f_i, g_i\}$ such that $m \mid U_i = f_i / g_i$ for each $i \in I$.

If $m \in \mathfrak{M}^*(\Omega)$ satisfies that

$$m \mid U_i = f_i / g_i \qquad (i \in I)$$

then $m^{-1} \in \mathfrak{M}^*(\Omega)$ is defined by

$$m^{-1} \mid U_i = g_i / f_i \qquad (i \in I)$$

Therefore $\mathfrak{M}^*(\Omega)$ is an abelian multiplicative group and thus $\mathfrak{M}^*(\Omega) \in \mathrm{Obj}(A_b)$.

For Ω and Ω' in $\mathrm{Obj}(O(C))$ and $f \in \mathrm{Morph}(O(C))$ such that $f: \Omega \longrightarrow \Omega'$

$$\mathfrak{M}^*(f) = f^* : \mathfrak{M}^*(\Omega') \longrightarrow \mathfrak{M}^*(\Omega)$$

is defined by $f^*(m') = m' \circ f \in \mathfrak{M}^*(\Omega)$, where $m' \in \mathfrak{M}^*(\Omega')$ and $\mathrm{Morph}(O(C))$ is the class of all morphisms of $O(C)$. It follows that $\mathfrak{M}^*$ is a contravariant functor.

Definition 2.1. For $\Omega \in \mathrm{Obj}(O(C))$, suppose that a function $p: \Omega \longrightarrow Z$ (Z = the set of all integers), is defined such that $\{z \in \Omega \mid p(z) \neq 0\}$ is a discrete subset of Ω. We call the formal sum

$$\sum_{z\in\Omega} p(z)\cdot z$$

a divisor on Ω. Moreover, if $d=\sum_{z\in\Omega} p(z)\cdot z$,

$\sum_{z\in\Omega} p(z)$ is called the degree of d, and we denote this by deg(d), i. e, $\deg(d)=\sum_{z\in\Omega} p$

Example 2.2. For $\Omega\in\mathrm{Obj}(\boldsymbol{O}(C))$, consider a meromorphic function $m\in\mathfrak{M}^*(\Omega)$. Then at each point $\zeta\in\Omega$, we have the Laurent expansion of m such that $m(z)=\sum_{j=N}^{\infty} a_j(z-\zeta)^j$, where $a_N\neq 0$. In this case, N is called the order of m at ζ and is denoted by $N=\mathrm{Ord}(m, \zeta)$. Clearly, if $m=f/g$ in some neighbourhood of ζ, then we have

$$\mathrm{Ord}(m, \zeta)=\mathrm{Ord}(f, \zeta)-\mathrm{Ord}(g, \zeta),$$

where f and g are analytic functions on some neighbourhood of ζ.
We define

$$\mathrm{div}(m)=\sum_{z\in\Omega}\mathrm{Ord}(m, z)\cdot z,$$

and we call div(m) the divisor of m. In this case, we can easily prove that

1°. $\mathrm{div}(mm')=\mathrm{div}(m)+\mathrm{div}(m')$, $\quad m, m'\in\mathfrak{M}^*(\Omega)$

2°. $\mathrm{div}(m^{-1})=-\mathrm{div}(m)$, $\quad m\in\mathfrak{M}^*(\Omega)$

3°. $\mathrm{div}(m)=0 \Longleftrightarrow m$ is a nowhere vanishing analytic function on Ω.

For $\Omega\in\mathrm{Obj}(\boldsymbol{O}(C))$ we put

$$\mathcal{D}(\Omega)=\text{the set of all divisors on } \Omega.$$

Then, $\mathcal{D}(\Omega)$ is an abelian group, since for divisors

$$d=\sum_{z\in\Omega} p(z)\cdot z\ , \quad d'=\sum_{z\in\Omega} p'(z)\cdot z$$

we can define

$$d+d'=\sum_{z\in\Omega}(p(z)+p'(z))\cdot z$$

which is also a divisor on Ω.

Proposition 2.3. We have a contravariant functor

$$\begin{array}{ccc} \mathcal{D}: \boldsymbol{O}(C) & \longrightarrow & A_b \\ \Cup & & \Cup \\ \Omega & \rightsquigarrow & \mathcal{D}(\Omega) \end{array}$$

Proof. For each $f: \Omega\to\Omega'\in\mathrm{Morph}(\boldsymbol{O}(C))$ we define

$$\mathcal{D}(f): \mathcal{D}(\Omega')\longrightarrow\mathcal{D}(\Omega)$$

by

$$\mathcal{D}(f)(d')=\sum_{z\in\Omega} p'(f(z))\cdot z$$

where

$$d'=\sum_{z\in\Omega'} p'(z)\cdot z\in\mathcal{D}(\Omega').$$

Then for each $g\circ f:\Omega\to\Omega'\to\Omega''\in Morph\ (\boldsymbol{O}(\boldsymbol{C}))$ we have

$$\mathcal{D}(g\circ f)=\mathcal{D}(f)\circ\mathcal{D}(g).$$

In fact, for each $d''=\sum_{z\in\Omega''}p''(z)\cdot z\in\mathcal{D}(\Omega'')$

$$\mathcal{D}(g\circ f)\ (d'')=\sum_{z\in\Omega}p''\ (g\circ f(z))\cdot z$$

and

$$\mathcal{D}(f)\circ\mathcal{D}(g)\ (d'')=\mathcal{D}(f)\ \ (\sum_{z'\in\Omega'}p''\ (g(z')\cdot z')=\sum_{z\in\Omega}p''\ (g\circ f(z))\cdot z$$

If $1_\Omega:\Omega\longrightarrow\Omega$ is the identity map in $Morph\ (\boldsymbol{O}(\boldsymbol{C}))$ it is clear that

$$\mathcal{D}(1_\Omega)=1_{\mathcal{D}(\Omega)}$$

Thus; $\mathcal{D}$ is a contravariant functor. ∎

Theorem 2.4. There exists a natural transformation

$$\mathrm{div}:\mathfrak{M}^*\longrightarrow\mathcal{D}$$

which is surjective.

Proof. Our proof is divided into three steps.

Step I : We shall prove that div is a natural transformation. For each $\Omega\in$ Obj $(\boldsymbol{O}(\boldsymbol{C}))$, we define $\mathrm{div}_\Omega:\mathfrak{M}^*(\Omega)\longrightarrow\mathcal{D}(\Omega)$

by

$$\mathrm{div}_\Omega(m)=\mathrm{div}\ (m)$$

for each $m\in\mathfrak{M}^*(\Omega)$. Consider $f:\Omega\longrightarrow\Omega'\in Morph\ (\boldsymbol{O}(\boldsymbol{C}))$, then we get the commutative diagram

$$\begin{array}{ccc} \mathfrak{M}^*(\Omega') & \xrightarrow{\ \mathrm{div}_{\Omega'}\ } & \mathcal{D}(\Omega') \\ {\scriptstyle \mathfrak{M}^*(f)}\big\downarrow & & \big\downarrow{\scriptstyle \mathcal{D}(f)} \\ \mathfrak{M}^*(\Omega) & \xrightarrow{\ \mathrm{div}_{\Omega}\ } & \mathcal{D}(\Omega) \end{array},$$

from the fact that for each $m'\in\mathfrak{M}^*(\Omega')$

$$\mathrm{div}_\Omega\circ\ \mathfrak{M}^*(f)\ (m')=\mathrm{div}_\Omega(m'\circ f)=\sum_{z\in\Omega}\mathrm{Ord}\,(m',\ f(z))\cdot z$$

and

$$\begin{aligned}\mathcal{D}(f)\circ\mathrm{div}_{\Omega'}\ (m')&=\mathcal{D}(f)\ (\sum_{z\in\Omega'}\mathrm{Ord}\,(m',z)\cdot z)\\ &=\sum_{z\in\Omega}\mathrm{Ord}\,(m',\ f(z))\cdot z\end{aligned}$$

That is, div : $\mathfrak{M}^* \to \mathcal{D}$ is a natural transformation.

Step II : Let $\mathscr{U} = \{U_i \mid i \in I\}$ be an open covering for Ω, where $\Omega \in \mathrm{Obj}(\boldsymbol{O}(\boldsymbol{C}))$. Suppose that we are given $m_i \in \mathfrak{M}^*(U_i)$ for each $i \in I$ such that at each point of $U_i \cap U_j$ m_i / m_j is an analytic function for all $i, j \in I$. In this step, we shall prove that there exists $m \in \mathfrak{M}^*(\Omega)$ such that m / m_i is analytic in Ω_i for all $i \in I$.

By taking a refinement of the open covering $\mathscr{U}$, we can assume without loss of generality that each $U_i \in \mathscr{U}$ is convex. On each $U_i \cap U_j$, we put $h_{ij} = m_i / m_j$ and fix a branch of $\log h_{ij}$. Define

$$c_{ijk} = \frac{1}{2\pi i}(\log h_{ij} + \log h_{jk} + \log h_{ki}),$$

then $c_{ijk} \in \boldsymbol{Z}$ since $h_{ij} h_{jk} h_{ki} \equiv 1$. It follows that on $U_i \cap U_j \cap U_k$ c_{ijk} is constant (Note that $U_i \cap U_j \cap U_k$ is convex and thus it is connected).

Let us assume that the branches of $\log h_{ij}$ can be chosen so that $c_{ijk} = 0$ for all $i, j, k \in I$. If we set $f_{ij} = \log h_{ij}$, since

i) f_{ij} is analytic on $U_i \cap U_j$,

ii) $h_{ij} = 1 / h_{ji}$ on $U_i \cap U_j$, and

iii) $c_{ijk} = 0$,

we see that the following hold :

$$f_{ij} + f_{jk} + f_{ki} = 0 \quad \text{on } U_i \cap U_j \cap U_k$$

$$f_{ij} = -f_{ji} \qquad \text{on } U_i \cap U_j.$$

Therefore there exists an analytic function g_i defined on U_i ($i \in I$) such that

$$\log h_{ij} = f_{ij} = g_j - g_i \qquad \text{on } U_i \cap U_j \ \ ([2]).$$

Put $a_i = \exp(g_i)$ on U_i, then a_i is a non-zero analytic function on U_i. Then, it is clear that

$$m_i a_i = m_j a_j \ \text{ on } U_i \cap U_j$$

since $a_j / a_i = h_{ij} = m_i / m_j$. If we define $m \in \mathfrak{M}^*(\Omega)$ by taking $m \mid U_i = m_i a_i$, then m satisfies our required conditions.

To complete our proof we have to show that we can choose the branches of $\log h_{ij}$ such that $c_{ijk} = 0$ for all $i, j, k \in I$.

At first, we observe that $\{c_{ijk}\}$ is a class in $H^2(\mathscr{U}, \boldsymbol{Z})$, the 2^{nd} cohomology group of $\mathscr{U}$ with values in $\boldsymbol{Z}$ (see § 4). In fact, for $i, j, k, l \in I$

$$2\pi i(c_{jkl} - c_{ikl} + c_{ijl} - c_{ijk}) = \log h_{jk} + \log h_{kl} + \log h_{lj} - (\log h_{ik} + \log h_{kl} + \log h_{li}) + \log h_{ij} + \log h_{jl} + \log h_{li} - (\log h_{ij} + \log h_{jk} + \log h_{ki}) = 0,$$

we used here $\log h_{ij} = -\log h_{ji}$.

On the other hand, since for each i $U_i \in \mathscr{U}$ is convex we know that $H^p(U_i, \boldsymbol{Z}) =$ f $p \neq 0$, where $H^p(U_i, \boldsymbol{Z})$ is the p-th singular cohomology group of U_i with coefficien n $\boldsymbol{Z}$.

This implies that $H^2(\Omega, \boldsymbol{Z}) = 0$.

Hence by Leray's theorem ([7] or see § 4).

$$H^2(\mathscr{U}, \boldsymbol{Z}) \cong H^2(\Omega, \boldsymbol{Z}) = 0.$$

Therefore $\{c_{ijk}\}$ is a coboundary and there exists integers η_{ij} for $i, j, k \in I$ such th

$$c_{ijk} = \eta_{ij} + \eta_{jk} + \eta_{ki}$$

f we define a new branch of $\log h_{ij}$ by subtracting $2\pi i \eta_{ij}$ from the original choice (Note that i in $2\pi i$ denotes $\sqrt{-1}$), then we have

$$c_{ijk} = 0,$$

or the new choice of branches and for all $i, j, k \in I$.

Step III. We shall prove that for each $\Omega \in \mathrm{Obj}(\boldsymbol{O}(\boldsymbol{C}))$

$$\mathrm{div}_\Omega : \mathfrak{M}^*(\Omega) \longrightarrow \mathscr{D}(\Omega)$$

s surjective. Taking $d \in \mathscr{D}(\Omega)$ such that

$$d = \sum_{z \in \Omega} p(z) \cdot z$$

If $p(z_i) \neq 0 \neq p(z_j)$ $(z_i, z_j \in \Omega)$, then we may assume without loss of generality hat $z_i \in U_i$, $z_j \in U_j$ and $z_i, z_j \notin U_i \cap U_j$.

Then there exist $m_i \in \mathfrak{M}^*(U_i)$ and $m_j \in \mathfrak{M}^*(U_j)$ such that

$$m_i(z) = \sum_{k=p(z_i)}^{\infty} a_k (z - z_i)^k, \qquad z \in U_i$$

$$m_j(z) = \sum_{k=p(z_j)}^{\infty} b_k (z - z_j)^k, \qquad z \in U_j$$

with $a_{p(z_i)} \neq 0 \neq b_{p(z_j)}$. Then, by step II there exists only one $m \in \mathfrak{M}^*(\Omega)$ such that m/m_i is analytic on U_i. We have to note that for each $z \in U_i \cap U_j$ m_i / m_j is a non-zero analytic function. By our definitions above it is obvious that $\mathrm{div}_\Omega(m) = d$. Hence

$$\mathrm{div} : \mathfrak{M}^* \longrightarrow \mathscr{D} \qquad \blacksquare$$

s surjective.

§ 3. Holomorphic Line Bundles

Definition 3.1. Let M be a Hausdorff and paracompact topological space. M is said to be an n-dimensional complex manifold if there is an atlas $\mathcal{A}(M)=\{(U_i, \phi_i) \; i\in I\}$ such that

i) $\{U_i \mid i\in I\}$ is an open covering of M,

ii) ϕ_i is a homeomorphism of U_i onto the open subset $\phi_i(U_i)$ of C^n for all $i\in I$, and

iii) for each $i, j\in I$

$$\phi_i\circ\phi_j^{-1} : \phi_j(U_{ij}) \longrightarrow \phi_i(U_{ij})$$

is a biholomorphic map, where $U_{ij}=U_i\cap U_j$.

Definition 3.2. Let M be a topological space, Hausdorff and paracompact. Then $\pi: E\to X$ is called an n-dimensional complex vector bundle if and only if the following conditions are satisfied :

i) E is a topological space and π is a continuous map.

ii) The fibre $\pi^{-1}(x)=E_x$ has the structure of a complex n-dimensional vector space for any $x\in X$.

iii) There exists an open covering $\{U_i \mid i\in I\}$ of X and homeomorphisms $\theta_i : \pi^{-1}(U_i) \longrightarrow U_i\times C^n$ such that for all $i\in I$ and $x\in U_i$ $\theta_i \mid E_x : E_x \longrightarrow x\times C^n$ is a complex linear map and onto isomorphism (θ_i are called trivializations of E).

There is an alternative description of complex vector bundle in terms of transition functions as follows. Under the situation of Definition 3.2 for each $i\in I$ and $x\in U_i$ we let

$$\theta_{ix} : E_x \longrightarrow C^n$$

denote the restriction of θ_i to E_x composed with the projection on C^n.

Define

$$\theta_{ij} : U_{ij} \longrightarrow GL(n, C)$$

by $\theta_{ij}(x)=\theta_{ix}\cdot\theta_{jx}^{-1}$ for $x\in U_{ij}=U_i\cap U_j$ and $i, j\in I$.

Then it is easy to verify that the θ_{ij} are continuous maps which satisfies the following cocycle conditions ; namely,

$$\theta_{ij}=\theta_{ji}^{-1} \quad , \quad \theta_{ij}\cdot\theta_{jk}=\theta_{ik}$$

The θ_{ij} are called transitive functions of the complex vector bundle E. There is an one-to-one corresponding between the set of transitive functions and the set of com plex vector bundles ([6]).

Let M be a complex manifold. A complex vector bundle $\pi: E \longrightarrow M$ is called a holomorphic vector bundle over M if

i) E is a complex manifold

ii) $\pi: E \longrightarrow M$ is a holomorpic map

iii) each trivialization $\theta_i: E \mid U_i \longrightarrow U_i \times C^m$ is a biholomorphic map.

Definition 3.3. Let M be an one dimensional complex manifold. If M is connected and if for each atlas $\mathscr{A}(M) = \{(U_i, \phi_i) \mid i \in I\}$ $\phi_i \circ \phi_j^{-1}$ is a biholomorphism of $\phi_j(U_{ij})$ with $\phi_i(U_{ij})$ $(i, j \in I)$, then M is called a Riemann surface.

A holomorphic vector bundle $\pi: L \longrightarrow M$ over M is said to be a holomorphic line bundle if $\pi: L \longrightarrow M$ is one-dimensional.

We denote by ***CRS*** the category of all compact Riemann surfaces and all non-zero analytic functions between compact Riemann surfaces.

Then we have a contravariant functor

$$\begin{array}{ccc} \mathscr{HLB}: \boldsymbol{CRS} & \longrightarrow & A_b \\ \Downarrow & & \Downarrow \\ M & \longmapsto & \mathscr{HLB}(M), \end{array}$$

Here $\mathscr{HLB}(M)$ is the set of all isomorphism classes of holomorphic line bundles over M. The abelion group structure of $\mathscr{HLB}(M)$ is defined as follows. For $[L]$, $[N] \in \mathscr{HLB}(M)$

$$[L] \cdot [N] = [L \otimes_C N], \quad [L]^{-1} = [L^*],$$

where L^* is the dual bundle of L. Furthermore, for $f: M \longrightarrow M' \in \text{Morph}(\boldsymbol{CRS})$ we have

$$\begin{array}{ccc} \mathscr{HLB}(f): \mathscr{HLB}(M') & \longrightarrow & \mathscr{HLB}(M) \\ \Downarrow & & \Downarrow \\ [L] & \longmapsto & [f^* L] \end{array} \qquad ([6]).$$

By the same way as in § 2, we get two contravariant functors:

$$\begin{array}{ccc} \mathfrak{M}_c^*: \boldsymbol{CRS} & \longrightarrow & A_b \\ \Downarrow & & \Downarrow \\ M & \longmapsto & \mathfrak{M}_c^*(M) \end{array} , \qquad \begin{array}{ccc} \mathscr{D}_c: \boldsymbol{CRS} & \longrightarrow & A_b \\ \Downarrow & & \Downarrow \\ M & \longmapsto & \mathscr{D}_c(M), \end{array}$$

Here $\mathfrak{M}_c^*(M)$, $\mathscr{D}_c(M)$ denote the set of all non-zero meromorphism functions defined on M, the set of all divisors on M, respectively.

Lemma 3.4. Let $M \in \mathrm{Obj}\,(\boldsymbol{CRS})$ and $d \in \mathcal{D}_c(M)$. If there exists a meromorphic function $m \in \mathfrak{M}_c^*(M)$ such that $\mathrm{div}\,(m) = d$ then $\deg\,(d) = 0$.

Proof. Assume that $\mathrm{div}\,(m) = d$ $(m \in \mathfrak{M}_c^*(M))$ and

$$\mathrm{div}\,(m) = \sum_{i=1}^{n} n_i z_i \quad (n_i \in \boldsymbol{Z},\ z_i \in M).$$

(Note that, since M is compact, $\{z \in M \mid \mathrm{Ord}\,(m, z) \neq 0\}$ is a finite set). Then, we can choose local charts (U_i, ϕ_i) of M containing z_i such that $\phi_i(U_i)$ contains $\bar{D}_1(0)$, where

$$\bar{D}_1 = \{z \in \boldsymbol{C} \mid |z| \leqq 1\}$$

We shall put such that

$$D_i = \phi_i^{-1}(\bar{D}_1(0)), \quad \gamma_i = \partial D_i = \phi_i^{-1}(\partial \bar{D}_1(0)),$$

where ∂D_i is the boundary of D_i. Without loss of generality, we may assume that the sets D_i are mutually disjoint.

Since m is analytic on $M' = M \setminus \bigcup_{i=1}^{n} D_i$ $\mathrm{Log}\, m$ is defined up to integer multiples. Thus, 1-form $\phi = d\,(\log m)$ ([7]) is well defined on M'. Using Stokes' theorem we have

$$\int_{M'} d\phi = \sum_{i=1}^{n} \int_{\gamma_i} \phi$$

Since $d\phi = d^2\,(\log m) = 0$ ([7]), we have

$$0 = \sum_{i=1}^{n} \int_{\gamma_i} \phi = \sum_{i=1}^{n} \int_{\gamma_i} \frac{\partial}{\partial z} \log m_i dz \quad (m_i = m\,\phi_i^{-1})$$

$$= \sum_{i=1}^{n} \int_{\gamma_i} \frac{m_i'}{m_i} dz \quad (m_i' = \frac{dm_i}{dz})$$

$$= \sum_{i=1}^{n} n_i \quad \text{(by the residue theorem)}$$

Therefore, $\deg\,(d) = 0$. ∎

Definition 3.5. Let L be a holomorphic line bundle on a Riemann surface M with transition functions θ_{ij} relative to the open covering $\{U_i \mid i \in I\}$ of M. A family $\{m_i \in \mathfrak{M}^*(U_i)\}$ $(i \in I)$ is called a meromorphic section m of L if for all $i, j \in I$

$$m_i = \theta_{ij} m_j \quad \text{on } U_{ij} = U_i \cap U_j.$$

We shall denote the set of all meromorphic sections of L by $\mathcal{M}(L)$. As before, we let $\mathcal{M}^*(L)$ denote the set of all non-zero meromorphic sections of L.

Proposition 3.6. There is a natural transformation

$$[\quad] : \mathcal{D}_c \longrightarrow \mathcal{HLB}.$$

Furthermore, for $M \in \boldsymbol{CRS}$ and $d \in \mathcal{D}_c(M)$ there exists $s\,(d) \in \mathcal{M}^*([d])$ such that div $(s(d))=d$. Moreover, $s(d)$ is uniquely determined up to multiplication by a non-zero complex number.

Proof. For each $M\in \mathrm{Obj}(\boldsymbol{CRS})$ we define

$$[\quad]_M : \mathcal{D}_c(M) \longrightarrow \mathcal{HLB}(M)$$

as follows. Take $d\in \mathcal{D}_c(M)$. Then d is uniquely determined by an open covering $\{U_i\}$ $(i\in I)$ of M and $m_i\in \mathfrak{M}^*(U_i)$ such that for all $i, j\in I$ m_i/m_j is a non-zero analytic function on $U_{ij}=U_i\cap U_j$ (See Theorem 2.4.).

Define

$$\theta_{ij}=m_i/m_j \quad \text{on } U_{ij},$$

then $\{\theta_{ij} \mid i, j\in I\}$ satisfies the cocycle conditions

$$\theta_{ij}=\theta_{ji}^{-1} \quad , \quad \theta_{ij}\,\theta_{jk}=\theta_{ik} .$$

Therefore $\{\theta_{ij}\}$ $(i, j\in I)$ determines a holomorphic line bundle over M which we denote by $[d]$. Hence

$$\begin{array}{ccc} [\quad]_M : \mathcal{D}_c(M) & \longrightarrow & \mathcal{HLB}(M) \\ \Downarrow & & \Downarrow \\ d & \rightsquigarrow & [d] \end{array}$$

Let L and L' be holomorphic line bundles over M with transition functions $\{\theta_{ij}\}$ and $\{\theta'_{ij}\}$, respectively relative to the open covering $\{U_i \mid i\in I\}$ of M. Then, the transition functions of $L\otimes_C L'$ relative to $\{U_i \mid i\in I\}$ are $\theta_{ij}\cdot\theta'_{ij}$ for all $i, j\in I$.

Suppose that for $d, d'\in \mathcal{D}_c(M)$ d and d' are uniquely specified by $m_i\in \mathfrak{M}^*(U_i)$ and $m'_i\in \mathfrak{M}^*(U_i)$ respectively, relative to the open covering $\{U_i \mid i\in I\}$ of M (Note that m_i/m_j and m'_i/m'_j are non-zero analytic on $U_{ij}=U_i\cap U_j$). Then

$$d=\sum_{i\in I}\Big(\sum_{z\in U_i}\mathrm{Ord}(m_i, z)\cdot z\Big)$$

$$d'=\sum_{i\in I}\Big(\sum_{z\in U_i}\mathrm{Ord}(m'_i, z)\cdot z\Big)$$

Therefore we have the following

$$d+d'=\sum_{i\in I}\Big(\sum_{z\in U_i}(\mathrm{Ord}(m_i, z)+\mathrm{Ord}(m'_i, z))\cdot z\Big)$$

$$=\sum_{i\in I}\Big(\sum_{z\in U_i}\mathrm{Ord}(m_i m'_i, z)\cdot z\Big) .$$

This means that $d+d'$ is uniquely specified by $m_i m'_i\in \mathfrak{M}^*(U_i)$ relative to $\{U_i \mid i\in I\}$. Therefore

$$[d+d']=[d]\otimes_C[d'] ,$$

and thus $[\quad]_M$ is a group homomorphism.

For each $f: M\longrightarrow M'\in \mathrm{Morph}(\boldsymbol{CRS})$ we shall prove that the following diagram is commutative.

$$\begin{array}{ccc} \mathcal{D}_c(M') & \xrightarrow{\quad [\quad]_{M'} \quad} & \mathcal{HLB}(M') \\ \mathcal{D}_c(f) \downarrow & & \downarrow \mathcal{HLB}(f) \\ \mathcal{D}_c(M) & \xrightarrow{\quad [\quad]_{M} \quad} & \mathcal{HLB}(M) \end{array}$$

Take $d' \in \mathcal{D}_c(M')$ and suppose that L' is a representation of $[d']$. If d' is specified by $m'_i \in \mathfrak{M}^*(U_i)$ relative to the open covering $\{U_i \mid i \in I\}$ of M', then $\mathcal{HLB}(f)([d']) = [f^*L']$ has $\{m'_i f / m'_j f \mid i, j \in I\}$ as its transition functions relative to the open covering $\{f^{-1}(U_i) \mid i \in I\}$ of M. On the other hand, $\mathcal{D}_c(f)(d')$ is specified by $m'_i f \in \mathfrak{M}^*(f^{-1}(U_i))$ relative to the open covering $\{f^{-1}(U_i) \mid i \in I\}$ of M. Hence if we put $\mathcal{D}_c(f)(d') = \tilde{d}' \in \mathcal{D}_c(M)$ then

$$[\tilde{d}'] = [f^*L]$$

Thus, $[\quad] : \mathcal{D}_c \longrightarrow \mathcal{HLB}$ is a natural transformation.

Next, we suppose that $d \in \mathcal{D}_c(M)$ is specified by $m_i \in \mathfrak{M}^*(U_i)$ relative to the open covering $\{U_i \mid i \in I\}$ of M. Then, the transition functions of $[d]$ is $\{\theta_{ij} = m_i / m_j \mid i, j \in I\}$, where θ_{ij} is analytic on U_{ij}.

Difine

$$s(d)_i = m_i \in \mathfrak{M}^*(U_i) \quad (i \in I),$$

then it is easy to prove that $\theta_{ij}\, s(d)_j = s(d)_i$

Hence $\{s(d)_i \mid i \in I\}$ determines a meromorphic section $s(d) \in \mathcal{M}^*([d])$.

In this case, it is obvious that $\mathrm{div}(s(d)) = d$, since for a non-zero complex number α

$$\theta_{ij} = m_i / m_j = \alpha m_i / \alpha m_j$$ ∎

Theorem 3.7. There is an exact sequence of contravariant functors

$$\mathfrak{M}_c^* \xrightarrow{\quad \mathrm{div} \quad} \mathcal{D}_c \xrightarrow{\quad [\quad] \quad} \mathcal{HLB} .$$

That is, for each $M \in \mathrm{Obj}\,(\boldsymbol{CRS})$ we have an exact sequence of abelian groups

$$\mathfrak{M}_c^*(M) \xrightarrow{\quad \mathrm{div}_M \quad} \mathcal{D}_c(M) \xrightarrow{\quad [\quad]_M \quad} \mathcal{HLB}(M).$$

Proof. It is sufficient to prove that a divisor $d \in \mathcal{D}_c(M)$ $(M \in \mathrm{Obj}\,(\boldsymbol{CRS}))$ is the divisor of a meromorphic function $m (\in \mathfrak{M}_c^*(M))$ if and only if $[d]$ is trivial as a holomorphic line bundle over M.

Assume that $d = \mathrm{div}(m)$, $m \in \mathfrak{M}_c^*(M)$. For an open covering $\{U_i \mid i \in I\}$ of M we put $m_i = m \mid U_i$ and $\theta_{ij} = m_i / m_j \equiv 1$.

Then m_i defines the divisor d and, since $\theta_{ij} \equiv 1$, $[d]$ is holomorphicallv trivial.

Conversely, suppose that $[d]$ is holomorphically trivial. Then there exists an non-zero analytic section s of $[d]$. For s_i, which are local representatives of s, $\theta_{ij}S_j=S_i$, where θ_{ij} are transition functions of $[d]$. Since m_i/m_j, where $m_i \in \mathfrak{M}^*(U_i)$ defines the divisors d, we have $m_i s_j = m_j s_i$ on $U_{ij}=U_i \cap U_j$. Therefore, we can define a meromorphic function $m \in \mathfrak{M}_c^*(M)$ such that $m \mid U_i = m_i/s_i$ (Note that s_i is analytic on U_i). Hence $\mathrm{div}(m)=d$. ∎

Corollary 3.8. We assume that for each $M \in \mathrm{Obj}\,(\boldsymbol{CRS})$ every holomorphic line bundle over M has a non-zero meromorphic section. Then

$$[\quad]: \mathcal{D}_c \longrightarrow \mathcal{HLB}$$

is surjective.

Proof. Our proof will be divided into three steps.

Step I. For $M \in \mathrm{Obj}\,(\boldsymbol{CRS})$ we consider a holomorphic line bundle L over M and $s, t \in \mathcal{M}^*(L)$. We shall prove that s/t determines a non-zero meromorphic function on M. Suppose that L has transition functions θ_{ij} relative to the open covering $\{U_i \mid i \in I\}$ of M. Let s_i and t_i be local representatives of s and t, respectively. Then, we have the following

$$\theta_{ij}S_j=S_i, \quad \theta_{ij}t_j=t_i \text{ on } U_{ij}=U_i \cap U_j,$$

for all $i, j \in I$. Therefore $s_j/t_j=s_i/t_i$ on U_{ij}. Thus we can define $m \in \mathfrak{M}_c^*(M)$ such that $m \mid U_i = s_i/t_i$.

Step II. Under the situation of step I, if we put $\deg(s)=\deg(\mathrm{div}(s))$, then $\deg(s)$ is independent of s and depends only on L.

In fact, it is clear that $\mathrm{div}(s/t)=\mathrm{div}(s)-\mathrm{div}(t)$ for $s, t \in \mathcal{M}^*(L)$. By step I $s/t \in \mathfrak{M}_c^*(M)$ and thus it follows from proposition 3.6 that $\deg(s/t)=0$.

Therefore $\deg(s)=\deg(t)$.

Step III. Under the situation of step II, we shall first prove that $[\mathrm{div}(s)] = [\mathrm{div}(t)]$ for $s, t \in \mathcal{M}^*(L)$.

Since $s/t \in \mathfrak{M}_c^*(M)$ for $s, t \in \mathcal{M}^*(L)$,

$$[\mathrm{div}(s/t)] = [\mathrm{div}(s)-\mathrm{div}(t)] = [\mathrm{div}(s)] \otimes_c [\mathrm{div}(t)]^{-1}$$

is holomorphicall trivial (See Theorem 3.7.) Therefore

$$[\mathrm{div}(s)]^* = [\mathrm{div}(t)]^* = [\mathrm{div}(t)]^{-1},$$

and thus $[\mathrm{div}(s)] = [\mathrm{div}(t)]$.

Now since for each $[L] \in \mathcal{HLB}(M)$ there exists a non-zero meromorphic section s of L, the map

$$\begin{array}{ccc} [\quad]_M : \mathcal{D}_c(M) & \longrightarrow & \mathcal{HLB}(M) \\ \cup\!\!\!\!\!\!\;\;\cup & & \cup\!\!\!\!\!\!\;\;\cup \\ \mathrm{div}(s) & \longmapsto & [\mathrm{div}(s)] = [L] \end{array}$$

is surjective. ∎

By Theorem 3.7. and Corollary 3.8 the following is obvious.

Corollary 3.9. Under the situation of Corollary 3.8, there is an exact sequence of contravariant functors :

$$\mathfrak{M}_c^* \longrightarrow \mathcal{D}_c \longrightarrow \mathcal{HLB} \longrightarrow 1.$$

§ 4. Sheaf Cohomology and $\mathcal{HLB}$

Definition 4.1. Let X and $\mathcal{F}$ be topological spaces, and let $\pi : \mathcal{F} \rightarrow X$ be a local homeomorphism. We say that $(\mathcal{F}, \pi, X)$ is a sheaf of rings on X if

i) the stalks $\mathcal{F}_x = \pi^{-1}(x)$ $(x \in X)$ have the structure of a ring, and

ii) the ring operators in $\mathcal{F}_x$ $(x \in X)$ are continuous in the topology of $\mathcal{F}$.

Example 4.2. Let X be a complex manifold, and let $O(X)$ be the set of all open subsets of X. For each $U \in O(X)$ $\mathcal{A}(U)$ denotes the set of all analytic functions defined on U.

Also, for each $x \in X$ we put

$$\mathcal{U}_x = \{U \in O(X) \mid x \in U\}$$

For $U, V \in \mathcal{U}_x$ and $f \in \mathcal{A}(U)$, $g \in \mathcal{A}(V)$ we define an equivalence relation "$\sim_x$" as follows ;

$$f \sim_x g \iff \exists W \in \mathcal{U}_x \quad \text{s.t.} \; f \mid W = g \mid W.$$

(Note that $W \subset U \cap V$)

We shall put

$$A(x) = \dot{\bigcup_{U \in \mathcal{U}_x}} \mathcal{A}(U),$$

and define $\mathcal{A}_x$ as

$$\mathcal{A}_x = A(x) / \sim_x .$$

For each $f \in A(x)$ we let f_x denote the equivalence class of f under $\sim_x$ in $\mathscr{A}_x$. We call f_x the germ of f at x. It is known that for each $x \in X$ $\mathscr{A}_x$ is a Neotherian unique factorization domain ([2]).

Let us put

$$\underline{\mathscr{A}} = \bigcup_{x \in X} \mathscr{A}_x \quad ,$$

and we shall topologise $\underline{\mathscr{A}}$ as follows. We take the collection of sets

$$\{f_x \mid x \in U \subset \boldsymbol{O}(X) \text{ and } f \in \mathscr{A}(U)\}$$

as a base of open sets for the topology of $\underline{\mathscr{A}}$. Define

$$\begin{array}{ccc} \pi : \underline{\mathscr{A}} & \longrightarrow & X \\ \cup & & \Psi \\ \mathscr{A}_x & \longmapsto & x \end{array} \quad ,$$

then $(\underline{\mathscr{A}}, \pi, X)$ is a sheaf of rings.

For each $U \in \boldsymbol{O}(X)$ we define

$\tilde{\mathscr{A}}(U)$ = the set of all continuous sections of $\underline{\mathscr{A}}$ over U.

Then it is easy to prove that the contravariant functor

$$\begin{array}{ccc} \tilde{\mathscr{A}} : \boldsymbol{O}(X) & \longrightarrow & \boldsymbol{R}\text{ing} \\ \Psi & & \Psi \\ U & \longmapsto & \underline{\mathscr{A}}(U) \end{array}$$

is a presheaf, where $\boldsymbol{R}$ing is the category consisting of all rings and all ring homomorphisms. Here, we have to note that the sheaf $\underline{\mathscr{A}}$ is isomorphic to the sheafification of the presheaf $\tilde{\mathscr{A}}$. Moreover, the sheafification of the presheaf

$$\begin{array}{ccc} \mathscr{A} : \boldsymbol{O}(X) & \longrightarrow & \boldsymbol{R}\text{ing} \\ \Psi & & \Psi \\ U & \longmapsto & \mathscr{A}(U) \end{array}$$

is isomorphic to $\underline{\mathscr{A}}$.

Moreover, if we put for $U \in \boldsymbol{O}(X)$

$$\mathscr{A}^*(U) = \text{the set of all units in } \mathscr{A}(U),$$

then we see that the contravariant functor

$$\begin{array}{ccc} \mathscr{A}^* : \boldsymbol{O}(X) & \longrightarrow & \boldsymbol{F}\text{ield} \\ \Psi & & \Psi \\ U & \longmapsto & \mathscr{A}^*(U) \end{array}$$

is a presheaf, where $\boldsymbol{F}$ield is the category consiting of all fields and all field homomorphisms. We set

$$\underline{\mathscr{A}}^* = \text{the sheafification of } \mathscr{A}^*$$

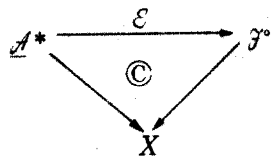
then $\underline{\mathscr{A}}^*$ is a sheaf of fields.

Next, we shall briefly describe on sheaf cohomology.

Let $(\underline{\mathscr{A}}^*)^\circ$ denote the sheaf of germs of sections not necessarily continuous, on $\underline{\mathscr{A}}^*$. Then there is the inclusion map

$$\mathscr{E} : \underline{\mathscr{A}}^* \longrightarrow \mathscr{J}^\circ \ (= (\underline{\mathscr{A}}^*)^\circ)$$

(Note that $\mathscr{E}$ is in fact a sheaf homomorphism ([2]) ; that is, in the commutative diagram



i) $\mathscr{E}$ is a continuous map

ii) for each $x \in X$

$$\mathscr{E} \mid \underline{\mathscr{A}}_x^* : \underline{\mathscr{A}}_x^* \longrightarrow \mathscr{J}_x^\circ$$

is a field homomorphism).

Set $\quad \mathscr{J}^1 = (\mathscr{J}^\circ / \underline{\mathscr{A}}^*)^\circ$,

which is the sheaf of germs of sections, not necessarily continuous on $\mathscr{J}^\circ / \underline{\mathscr{A}}^*$.

Proceeding inductively, let

$$\mathscr{J}^{k+1} = (\mathscr{J}^k / d_{k-1}(\mathscr{J}^{k-1}))^\circ \ ,$$

where $\mathscr{J}^{-1} = \underline{\mathscr{A}}^*$, $d_0 = \mathscr{E}$, and $d_{k-1} : \mathscr{J}^{k-1} \longrightarrow \mathscr{J}^k$. Then we have short exact sequences

$$0 \longrightarrow \underline{\mathscr{A}}^* \xrightarrow{\ \mathscr{E}\ } \mathscr{J}^\circ \xrightarrow{\ p\ } \mathscr{J}^\circ / \underline{\mathscr{A}}^* \longrightarrow 0$$

$$0 \longrightarrow \mathscr{J}^\circ / \underline{\mathscr{A}}^* \xrightarrow{\ \mathscr{E}_1\ } \mathscr{J}^1 \xrightarrow{\ p_1\ } \mathscr{J}^1 / (\mathscr{J}^\circ / \underline{\mathscr{A}}^*) \longrightarrow 0$$

$$0 \longrightarrow \mathscr{J}^k / \mathscr{J}^{k-1} \xrightarrow{\ \mathscr{E}_{k+1}\ } \mathscr{J}^{k+1} \xrightarrow{\ p_{k+1}\ } \mathscr{J}^{k+1} / (\mathscr{J}^k / \mathscr{J}^{k-1}) \longrightarrow 0$$

----

where $\mathscr{E}_i$ are inclusions, p_i are projections and

$$d_k = \mathscr{E}_{k+1} \circ p_k : \mathscr{J}^k \longrightarrow \mathscr{J}^{k+1} \ .$$

Consequently we have a long exact sequence of sheaves

$$0 \longrightarrow \underline{\mathscr{A}}^* \xrightarrow{\ \mathscr{E}\ } \mathscr{J}^\circ \xrightarrow{\ d_0\ } \mathscr{J}^1 \xrightarrow{\ d_1\ } \cdots\cdots$$

which is called the <u>canonical</u> <u>resolution</u> of $\underline{\mathscr{A}}^*$.

Since sheaf functors are left exact

$$0 \longrightarrow \underline{\mathscr{A}}^*(X) \xrightarrow{\ \mathscr{E}^*\ } \mathscr{J}^\circ(X) \xrightarrow{\ d_0^*\ } \mathscr{J}^1(X) \xrightarrow{\ d_1^*\ } \cdots\cdots$$

is semi-exact. We define

$$H^{\circ}(X, \underline{\mathscr{A}}^*) = \ker d_0^* \cong \underline{\mathscr{A}}^*(X)$$

$$H^k(X, \underline{\mathscr{A}}^*) = \ker d_k^* / I_m(d_{k-1}^*) \qquad (k>0)$$

Then $H^k(X, \underline{\mathscr{A}}^*)$ is an abelian group and is called the k-th sheaf cohomology group of X with coefficients in $\underline{\mathscr{A}}^*$.

Definition 4.3. Let ***CM*** be the category of all complex manifolds and all analytic functions between complex manifolds.

We denote the sheaf defined in Example 4.2 on $X \in \mathrm{Obj}\,(CM)$ by $\underline{\mathscr{A}}_X^*$

Let us define

$$\begin{array}{ccc} \mathscr{SH} : CM & \longrightarrow & A_b \\ \Uparrow & & \Uparrow \\ X & \rightsquigarrow & \mathscr{SH}(X) = H^1(X, \underline{\mathscr{A}}_X^*) \end{array} ,$$

then $\mathscr{SH}$ is a contravariant functor.

In fact, for each

$f : X \longrightarrow X' \in \mathrm{Morph}\,(CM)$ there exists a homomorphism f^* between the canonical flabby resolution of $\mathscr{A}_{X'}^*$ and $\mathscr{A}_X^*$ such that

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathscr{A}_{X'}^* & \xrightarrow{\ \mathcal{E}_{X'}\ } & \mathscr{F}_{X'}^{\circ} & \xrightarrow{\ d_0^{X'}\ } & \mathscr{F}_{X'}^1 & \longrightarrow & \cdots\cdots \\ & & f_{-1}^* \downarrow & & f_0^* \downarrow & & f_1^* \downarrow & & \\ 0 & \longrightarrow & \mathscr{A}_X^* & \xrightarrow{\ \mathcal{E}_X\ } & \mathscr{F}_X^{\circ} & \xrightarrow{\ d_0{}^X\ } & \mathscr{F}_X^1 & \longrightarrow & \cdots\cdots \end{array}$$

is commutative. Therefore, we can induce the abelian group homomorphism

$$\mathscr{SH}(f) : \mathscr{SH}(X') = H^1(X', \underline{\mathscr{A}}_{X'}^*) \longrightarrow \mathscr{SH}(X) = H^1(X, \underline{\mathscr{A}}_X^*),$$

which is natural, from $f^* = \{f_{-1}^*, f_0^*, \cdots\cdots\cdots\}$.

For each $X \in \mathrm{Obj}\,(CM)$ let $\mathscr{U} = \{U_i \mid i \in I\}$ be an open covering of X.

Let p be a non-negative integer. Given $s = (s_0, s_1, \cdots\cdots, s_p)$ we set $U_s = U_{s_0} \cap \cdots\cdots \cap U_{s_p}$, where $s_0, \cdots, s_p \in I$. A p-cochain of $\mathscr{U}$ with value in $\mathscr{A}_X^*$ is a map c which assigns to each $s \in I^{p+1}$ a section $c_s \in \underline{\mathscr{A}}_X^*(U_s)$ and for which c_s is an alternating function of s. That is,

$$c_{s_0 \cdots\cdots s_i \cdots\cdots s_j \cdots\cdots s_p} = -c_{s_0 \cdots\cdots s_j \cdots\cdots s_i \cdots\cdots s_p}$$

for $0 \leqq i \leqq j \leqq p$. put

$C^p(\mathscr{U}, \underline{\mathscr{A}}_X^*)$ = the abelian groups consisting of all p-cochains of $\mathscr{U}$ with value in $\underline{\mathscr{A}}_X^*$,

and define $D_p : C^p(\mathscr{U}, \underline{\mathscr{A}}_X^*) \longrightarrow C^{p+1}(\mathscr{U}, \underline{\mathscr{A}}_X^*)$ by

$$(D_p c)_s = \sum_{j=0}^{p+1} (-1)^j c_{s_0 \dots\dots \hat{s}_j \dots\dots s_{p+1}}, \quad s \in I^{p+2}$$

where $\hat{s}_j$ means that it is omitted. It follows that the sequence of abelian groups.

$$0 \longrightarrow C^0(\mathscr{U}, \mathscr{A}_X^*) \xrightarrow{D_0} C^1(\mathscr{U}, \mathscr{A}_X^*) \xrightarrow{D_1} \cdots\cdots$$

is semi-exact. Define

$$H^p(\mathscr{U}, \mathscr{A}_X^*) = \text{Ker } D_p / I_m D_{p-1}$$

where $H^0(\mathscr{U}, \mathscr{A}_X^*) = \text{Ker } D_0$. We call this abelian group the p-th cohomology group of $\mathscr{U}$ with value in $\mathscr{A}_X^*$.

Let $\mathscr{U}$ and $\mathscr{V}$ be open covering of X, and let $\mathscr{V}$ be a refinement of $\mathscr{U}$. There exists a canonical homomorphism

$$\gamma(\mathscr{U}, \mathscr{V}) : H^p(\mathscr{U}, \mathscr{A}_X^*) \longrightarrow H^p(\mathscr{V}, \mathscr{A}_X^*)$$ satisfying the following :

1° $\gamma(\mathscr{U}, \mathscr{U}) = 1$ on $H^p(\mathscr{U}, \mathscr{A}_X^*)$, and

2° for a refinement $\mathscr{W}$ of $\mathscr{V}$

$$\gamma(\mathscr{V}, \mathscr{W}) \cdot \gamma(\mathscr{U}, \mathscr{V}) = \gamma(\mathscr{U}, \mathscr{W}).$$

Therefore, for a non-zero integer p $\{H^p(\mathscr{U}, \mathscr{A}_X^*), \ \gamma(\mathscr{U}, \mathscr{V})\}$ forms a direct system. We define

$$\check{H}^p(X, \mathscr{A}_X^*) = \varinjlim_{\mathscr{U}} H^p(\mathscr{U}, \mathscr{A}_X^*),$$

and call it the p-th Čech cohomology group of X with coefficients in $\mathscr{A}_X^*$. Leray's theorem says that there exists an isomorphism

$$\chi : \check{H}^p(X, \mathscr{A}_X^*) \longrightarrow H^p(X, \mathscr{A}_X^*),$$

which is natural ([2]) (Note that $X \in \text{Obj}(\boldsymbol{CM})$).

Theorem 4.4. There is a natural transformation

$$\psi : \mathscr{SH} \longrightarrow \mathscr{HLB},$$

which is an isomorphism, where

$$\begin{array}{ccc} \mathscr{HLB} : \boldsymbol{CM} & \longrightarrow & A_b \\ \Cup & & \Cup \\ X & \rightsquigarrow & \mathscr{HLB}(X) \end{array}$$

is defined by the same way as in §3.

Proof. It is sufficient to prove that for each $X \in \text{Obj}(\boldsymbol{CM})$

$$\psi_X : \mathscr{SH}(X) \longrightarrow \mathscr{HLB}(X)$$

is a natural isomorphism. Recall that there is a natural isomorphism

$$\chi : \check{H}^1(X, \mathscr{A}_X^*) \longrightarrow H^1(X, \mathscr{A}_X^*) = \mathscr{SH}(X)$$

Hence for each $\xi \in \mathcal{SH}(X)$ we may find an open covering $\mathcal{U} = \{U_i \mid i \in I\}$ of X and $\{\phi_{ij}\} = \mathrm{Ker}\ D_1$ such that the canonical homomorphism

$$\chi^{-1}(\mathcal{U}) : H^1(\mathcal{U}, \mathcal{A}_X^*) \longrightarrow \check{H}^1(X, \mathcal{A}_X^*)$$

maps the cohomology class of $\{\phi_{ij}\}$ to $\chi^{-1}(\xi)$. The cocycle condition on $\{\phi_{ij}\}$ imply that $\phi_{ij}\phi_{jk} = \phi_{ik}$ for all $i, j, k \in I$. Since $\phi_{ij} : U_{ij} = U_i \cap U_j \longrightarrow GL(1, C)$ is analytic, we see that the ϕ_{ij} are the transition functions for a holomorphic line bundle $L(\xi)$ over X. We can easily prove that $L(\xi)$ depends only on ξ and not on our choice of covering or cocycle.

Therefore

$$\begin{array}{rcl} \psi_X : \mathcal{SH}(X) & \longrightarrow & \mathcal{HLB}(X) \\ \Psi & & \Psi \\ \xi & \rightsquigarrow & [L(\xi)] \end{array}$$

is a natural homomorphism. In particular it is obvious that ψ_X is an isomorphism (See proofs of Theorem 2.4 and Theorem 3.7). ▮

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