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# A NOTE ON A CLASS OF STARLIKE FUNCTIONS

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## 1. Introduction

Let $\mathcal{S}$ denote the class of functions

$$f(z)=z+\sum_{n=2}^{\infty} a_n z^n$$

which are analytic and univalent in the unit disk $\mathcal{U}=\{z: |z|<1\}$. Then a function $f(z)\in\mathcal{S}$ is said to be starlike with respect to the origin in the unit disk $\mathcal{U}$ if and only if

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\}>0 \qquad (z\in\mathcal{U}).$$

We denote by $\mathcal{S}^*$ the class of all starlike functions $f(z)$ in the unit disk $\mathcal{U}$. Further a function $f(z)\in\mathcal{S}$ is said to be starlike of order $k(0\leqq k\leqq 1)$ with respect to the origin in the unit disk $\mathcal{U}$ if and only if

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\}>k \qquad (z\in\mathcal{U})$$

for some $k(0\leqq k\leqq 1)$. And we denote by $\mathcal{S}^*(k)$ the class of all such functions $f(z)$. This class $\mathcal{S}^*(k)$ was first introduced by M. S. Robertson [2] and has been studied by E. P. Merkes [3] and A. Schild [4].

In the present paper we consider the class $\mathcal{S}(\alpha)$ $(1/2\leqq\alpha\leqq 1)$ of functions $f(z)\in\mathcal{S}$ satisfying the following condition

$$(1) \qquad \left|\frac{zf'(z)}{f(z)}-\alpha\right|\leqq\alpha \qquad (z\in\mathcal{U})$$

for some $\alpha$ $(1/2\leqq\alpha\leqq 1)$. This class $\mathcal{S}(\alpha)$ was introduced by R. Singh [5]. Furthermore R. Singh [6] showed some results for the class $\mathcal{S}(1)$. We can see that $\mathcal{S}(\alpha)\subset\mathcal{S}^*$ for any $\alpha$ $(1/2\leqq\alpha\leqq 1)$.

## 2. An argument theorem

At first, we need the following lemma was obtained by B. Pinchuk [2].

LEMMA. *Let a function*

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$$f(z)=z+\sum_{n=2}^{\infty} a_n z^n$$

*be in the class* $\mathcal{S}^*(k)$. *Then*

$$\left|\arg\left(\frac{f(z)}{z}\right)\right| \leqq 2(1-k)\sin^{-1}|z| \tag{2}$$

*for* $z\in U$.

Now we show the following theorem for the function $f(z)$ belonging to the class $\mathcal{S}(\alpha)$ with the aid of Lemma.

THEOREM. *Let a function*

$$f(z)=z+\sum_{n=2}^{\infty} a_n z_n$$

*be in the class* $\mathcal{S}(\alpha)$. *Then we have*

$$|\arg\{f'(z)\}| \leqq 2\sin^{-1}|z|+\sin^{-1}\left(\frac{(2\alpha-1)|z|}{\alpha-(1-\alpha)|z|^2}\right) \tag{3}$$

*for* $z\in\mathcal{U}$.

*Proof*. Let

$$g(z)=\frac{zf'(z)}{\alpha f(z)}-1, \tag{4}$$

then $|g(z)|\leqq 1$ and $g(0)=1/\alpha-1$. Further let

$$\begin{aligned} h(z)&=\frac{g(z)-g(0)}{1-g(0)g(z)} \\ &=\frac{zf'(z)/f(z)-1}{\alpha-(1-\alpha)(zf'(z)/\alpha f(z)-1)}, \end{aligned} \tag{5}$$

then $|h(z)|\leqq 1$ for $z\in\mathcal{U}$ and $h(0)=0$. Hence, by using Schwarz's lemma, we can write $h(z)=z\phi(z)$, where $\phi(z)$ is an analytic function in the unit disk $\mathcal{U}$ and satisfies $|\phi(z)|\leqq 1$ for $z\in\mathcal{U}$. Consequently we obtain

$$\begin{aligned} \frac{zf'(z)}{f(z)}&=\frac{\alpha(1+h(z))}{\alpha+(1-\alpha)h(z)} \\ &=\frac{\alpha(1+z\phi(z))}{\alpha+(1-\alpha)z\phi(z)} \end{aligned} \tag{6}$$

After a simple computation, we have

$$\left|\frac{zf'(z)}{f(z)}-\frac{\alpha(\alpha-(1-\alpha)|z|^2)}{\alpha^2-(1-\alpha)^2|z|^2}\right| \leqq \frac{\alpha(2\alpha-1)|z|}{\alpha^2-(1-\alpha)^2|z|^2} \tag{7}$$

which gives that

$$\left|\arg\left\{\frac{zf'(z)}{f(z)}\right\}\right| \leqq \sin^{-1}\left(\frac{(2\alpha-1)|z|}{\alpha-(1-\alpha)|z|^2}\right) \tag{8}$$

Now, since $f(z)\in\mathcal{S}(\alpha)$, $f(z)$ belongs to the class $\mathcal{S}^*=\mathcal{S}^*(0)$. Therefore

(9) $$\left|\arg\left(\frac{f(z)}{z}\right)\right|\leqq 2\sin^{-1}|z| \qquad (z\in\mathcal{U})$$

with the aid of (2) of Lemma. By using (8) and (9), we get

(10) $$\begin{aligned}|\arg\{f'(z)\}|&\leqq\sin^{-1}\left(\frac{(2\alpha-1)|z|}{\alpha-(1-\alpha)|z|^2}\right)+\left|\arg\left(\frac{f(z)}{z}\right)\right|\\&\leqq 2\sin^{-1}|z|+\sin^{-1}\left(\frac{(2\alpha-1)|z|}{\alpha-(1-\alpha)|z|^2}\right)\end{aligned}$$

for $z\in\mathcal{U}$. This completes the proof of the theorem.

COROLLARY 1. *Let a function*

$$f(z)=z+\sum_{n=2}^{\infty}a_nz^n$$

*be in the class* $\mathcal{S}(1)$. *Then we have*

(11) $$|\arg\{f'(z)\}|\leqq 3\sin^{-1}|z|$$

*for* $z\in\mathcal{U}$.

COROLLARY 2. *Let a function*

$$f(z)=z+\sum_{n=2}^{\infty}a_nz^n$$

*be in the class* $\mathcal{S}(1/2)$. *Then we have*

(12) $$|\arg\{f'(z)\}|\leqq 2\sin^{-1}|z|+\sin^{-1}0$$

*for* $z\in U$.

## References


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