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# CONFORMAL CHANGE IN 2-DIMENSIONAL UNIFIED FIELD THEORY

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## I. Introduction

IA. *Two dimensional unified field theory* (2-*g*-*UFT*): In the usual Einstein's 2-*g*-*UFT* the generalized 2-dimensional Riemannian space $X_2$, referred to a real coordinate system $x^\nu$, is endowed with a real nonsymmetric tensor $g_{\lambda\mu}$ which may be split into its symmetric part $h_{\lambda\mu}$ and skew-symmetric part $k_{\lambda\mu}$(*):

(1.1)a $$g_{\lambda\mu}=h_{\lambda\mu}+k_{\lambda\mu},$$

where

(1.1)b $$\mathfrak{g}=\mathrm{Det}(g_{\lambda\mu})\neq 0,\quad \mathfrak{h}=\mathrm{Det}(h_{\lambda\mu})\neq 0,$$
$$\mathfrak{k}=\mathrm{Det}(k_{\lambda\mu})=(k_{12})^2\neq 0.$$

We may define a unique tensor $h^{\lambda\nu}$ by

(1.2) $$h_{\lambda\mu}h^{\lambda\nu}=\delta_\mu^{\ \nu}.$$

In 2-*g*-*UFT* we use both $h_{\lambda\mu}$ and $h^{\lambda\nu}$ as tensors for raising and/or lowering indices of all tensors defined in $X_2$ in the usual manner.

The densities defined in (1.1)b are related by

(1.3)a $$\mathfrak{g}=\mathfrak{h}+\mathfrak{k},$$

so that

(1.3)b $$g=1+k,$$

where

(1.3)c $$g=\mathfrak{g}/\mathfrak{h},\quad k=\mathfrak{k}/\mathfrak{h}.$$

The following tensors will be used in our further considerations:

(1.3)d $$^{(0)}k_\lambda^{\ \nu}=\delta_\lambda^{\ \nu},\quad ^{(p)}k_\lambda^{\ \nu}={}^{(p-1)}k_\lambda^{\ \alpha}k_\alpha^{\ \nu},\quad (p=1,2,\cdots).$$

The differential geometric structure is imposed on $X_2$ by the tensor $g_{\lambda\mu}$ by means of a connection $\Gamma_{\lambda\mu}^{\ \ \nu}$ given by the following system of Einstein's equations

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(*) Throughout the present paper, all indices take values 1, 2 and follow the summation convention with the exception of indices $x, y, z$. Greek indices are used for the holonomic components of a tensor and Roman indices for the nonholonomic components.

(1.4) $$D_\omega g_{\lambda\mu}=2S_{\omega\mu}{}^{\alpha}g_{\lambda\alpha},$$

where $D_\omega$ is the symbolic vector of the covariant derivative with respect to $\Gamma_{\lambda\mu}{}^{\nu}$ and

(1.5) $$S_{\lambda\mu}{}^{\nu}=\Gamma_{[\lambda\mu]}{}^{\nu}.$$

We note from the last condition of (1.1)b that *there exists only the first class of $k_{\lambda\mu}$ in 2-g-UFT.*

IB. *Purpose.* The conformal change of $g_{\lambda\mu}$ was primarily studied in $X_4$ by Hlavaty ([1], p. 151). The purpose of the present paper is to investigate how the conformal change enforces the connection in 2-*g*-*UFT* and to give the complete relations between connections in terms of $g_{\lambda\mu}$.

## II. Recurrence relations in $X_2$

In this section, the recurrence relations in $X_2$, obtained by Chung ([2]), will be briefly introduced without proof. These relations will be needed in our further considerations.

Throughout the present paper we use the following Mishra's abbreviations, denoting an arbitrary tensor $T_{\omega\mu\nu}$ by $T$([3]):

(2.1)a $$\overset{xyz}{A}{}_{\omega\mu\nu}^{\alpha\beta\gamma}={}^{(x)}K_{\omega}{}^{\alpha}\,{}^{(y)}K_{\mu}{}^{\beta}\,{}^{(z)}K_{\nu}{}^{\gamma},$$

(2.1)b $$\overset{pqr}{T}=\overset{pqr}{T}_{\omega\mu\nu}=\overset{pqr}{A}{}_{\omega\mu\nu}^{\alpha\beta\gamma}T_{\alpha\beta\gamma},\quad T=T_{\omega\mu\nu}=\overset{000}{T}.$$

If the tensor $\overset{pqr}{T}_{\omega\mu\gamma}$ is skew-symmetric in the first two indices, then we have

(2.2) $$\overset{pqr}{T}_{\omega\mu\nu}=-\overset{qpr}{T}_{\mu\omega\nu}.$$

IIA. *The first recurrence relations.* We have

(2.3) $$^{(p+2)}k_{\lambda}{}^{\nu}+k\,^{(p)}k_{\lambda}{}^{\nu}=0,\quad (p=0,1,2,\cdots).$$

IIB. *The second recurrence relations.* If $T_{\omega\mu\nu}$ is a tensor skew-symmetric in the first two indices, we have

(2.4)a $$\overset{(10)r}{T}=0,$$

$$(r=0,1,2,\cdots)$$

(2.4)b $$\overset{11r}{T}=k\overset{00r}{T}.$$

IIC. *The third recurrence relations.* If $T_{\omega\mu\nu}$ is a tensor skew-symmetric in the first two indices, then we have

(2.5)a $$\overset{r(10)}{T}_{\nu[\omega\mu]}=0,$$

$$(r=0,1,2,\cdots)$$

(2.5)b $$\overset{r11}{T}_{\nu[\omega\mu]}=k\overset{r00}{T}_{\nu[\omega\mu]}.$$

IID. If $T_{\omega\mu\nu}$ is a tensor skew-symmetric in the first two indices, then we have

$$T_{[\omega\mu\nu]}=0. \tag{2.6}$$

## III. Conformal Change in 2-*g*-*UFT*

Consider two spaces $X_2$ $(\bar{X}_2)$, on which the differential geometric structure is imposed by a general real tensor $g_{\lambda\mu}$ $(\bar{g}_{\lambda\mu})$ through the connection $\Gamma_{\lambda\mu}{}^{\nu}$ $(\bar{\Gamma}_{\lambda\mu}{}^{\nu})$ defined respectively (1.4) and

$$\bar{D}_{\omega}\bar{g}_{\lambda\mu}=2\bar{S}_{\omega\mu}{}^{\alpha}\bar{g}_{\lambda\alpha}. \tag{3.1}$$

REMARK. In our subsequent considerations, we agree that, if $T$ is a function of $g_{\lambda\mu}$, then we denote by $\bar{T}$ the same function of $\bar{g}_{\lambda\mu}$. In particular, if $T$ is a tensor, so is $\bar{T}$. Furthermore, the indices of $T$ (of $\bar{T}$) will be raised and/or lowered by means of $h_{\lambda\mu}$ and/or $h^{\lambda\nu}$ (by means of $\bar{h}_{\lambda\mu}$ and/or $\bar{h}^{\lambda\nu}$).

We say that $X_2$ and $\bar{X}_2$ are *conformal* if and only if

$$\bar{g}_{\lambda\mu}(x)=\exp[\Omega(x)]g_{\lambda\mu}(x), \tag{3.2}$$

where $\Omega=\Omega(x)$ is an arbitrary function of position with at least two derivatives. This conformal change enforces a change of the connection, and it can always be expressed as follows ([1], p. 151):

$$\bar{\Gamma}_{\lambda\mu}{}^{\nu}=\Gamma_{\lambda\mu}{}^{\nu}+Q^{\nu}{}_{\lambda\mu}, \tag{3.3}$$

or equivalently

$$\bar{\Gamma}_{\lambda\mu}{}^{\nu}=\Gamma_{\lambda\mu}{}^{\nu}+M_{\lambda\mu}{}^{\nu}+N^{\nu}{}_{\lambda\mu}, \tag{3.4}$$

where

$$M_{\lambda\mu}{}^{\nu}=Q_{(\lambda\mu)}{}^{\nu},\quad N^{\nu}{}_{\lambda\mu}=Q_{(\lambda\mu)}{}^{\nu}. \tag{3.5}$$

The main purpose of the present paper is to express the tensors $M_{\lambda\mu}{}^{\nu}$ and $N^{\nu}{}_{\lambda\mu}$ in terms of $g_{\lambda\mu}$.

THEOREM 3.1. *We have*

$$\Omega_{\omega}h_{\lambda\mu}=2N_{[\lambda\mu]\omega}+2\overset{001}{M}_{\omega[\mu\lambda]}, \tag{3.6a}$$

$$\Omega_{\omega}k_{\lambda\mu}=2M_{\omega[\mu\lambda]}+2\overset{100}{N}_{[\lambda\mu]\omega}, \tag{3.6b}$$

*where* $\Omega_{\omega}=\partial_{\omega}\Omega$.

*Proof.* A simple calculation based on (3.2) and (3.4) gives

$$\begin{aligned}e^{-\Omega}(\bar{D}_{\omega}\bar{g}_{\lambda\mu}-2\bar{S}_{\omega\mu}{}^{\alpha}\bar{g}_{\lambda\alpha})&=D_{\omega}g_{\lambda\mu}-2S_{\omega\mu}{}^{\alpha}g_{\lambda\alpha}+\Omega_{\omega}h_{\lambda\mu}-2\overset{001}{M}_{\omega(\mu\lambda)}\\&\quad-2N_{(\lambda\mu)\omega}+\Omega_{\omega}k_{\lambda\mu}-2M_{\omega[\mu\lambda]}-2\overset{100}{N}_{[\lambda\mu]\omega}.\end{aligned}$$

Our assertions follow from above relation by means of (1.4) and (3.1).

THEOREM 3.2. *The tensors $a_{\omega\mu\lambda}$ and $b_{\omega\mu\lambda}$, defined by*

(3.7)a $$a_{\omega\mu\lambda}=\Omega_\omega h_{\lambda\mu}+\Omega_\mu h_{\omega\lambda}-\Omega_\lambda h_{\mu\omega},$$

(3.7)b $$b_{\omega\mu\lambda}=\Omega_\omega k_{\lambda\mu}+\Omega_\mu k_{\omega\lambda}-\Omega_\lambda k_{\mu\omega},$$

*may be given by*

(3.8)a $$a_{\omega\mu\lambda}=2N_{\lambda\omega\mu}-4\overset{001}{M}_{\lambda[\omega\mu]},$$

(3.8)b $$b_{\omega\mu\lambda}=2M_{\omega\mu\lambda}+4\overset{100}{N}_{[\omega\mu]\lambda}.$$

*Proof.* (3.8) are the results of (3.6), (3.7), and (2.4).

THEOREM 3.3. *The equations* (3.8) *are equivalent to*

(3.9)a $$M_{\omega\mu\lambda}=\frac{1}{2g}H_{\omega\mu\lambda},$$

(3.9)b $$N_{\lambda\omega\mu}=\frac{1}{2}a_{\omega\mu\lambda},$$

*where*

(3.10) $$H_{\omega\mu\lambda}=b_{\omega\mu\lambda}-2\overset{001}{a}_{\lambda[\omega\mu]}=2\Omega_\alpha k_{[\omega}{}^\alpha h_{\mu]\lambda}-\Omega_\lambda k_{\omega\mu}.$$

*Proof.* The last part of (3.10) follows from (3.7). Eliminating $N_{\omega\mu\lambda}$ from (3.8) and using (2.4) and (2.5), we have

(3.11) $$2M_{\omega\mu\lambda}-4kM_{\lambda[\omega\mu]}=H_{\omega\mu\lambda}.$$

(3.9)a may be obtained from (3.11) by means of (1.3)b and the following relation equivalent to (2.6):

(3.12) $$2M_{\lambda[\omega\mu]}=-M_{\omega\mu\lambda}.$$

The second relation (3.9)b may be obtained by substituting (3.9)a for $M_{\omega\mu\lambda}$ into (3.8)a.

Now that we have found the tensors $M_{\omega\mu}{}^\lambda$ and $N^\lambda{}_{\omega\mu}$ in terms of $g_{\lambda\mu}$, it is possible to complete the relation (3.4) as in the following main theorem:

THEOREM 3.4. *Under the conformal change* (3.2), *the Einstein's connections $\bar{\Gamma}_{\omega\mu}{}^\lambda$ and $\Gamma_{\omega\mu}{}^\lambda$ are related as*

(3.13) $$\bar{\Gamma}_{\omega\mu}{}^\lambda=\Gamma_{\omega\mu}{}^\lambda+\frac{1}{2g}(\Omega_\beta k_{\mu\omega}h^{\beta\lambda}+2\Omega_\alpha k_{[\omega}{}^\alpha h_{\mu]}{}^\lambda$$
$$+g\Omega_{(\omega}h_{\mu)}{}^\lambda-g\Omega_\alpha h_{\mu\omega}h^{\lambda\alpha}).$$

*Proof.* Substituting (3.9) into (3.4), we have (3.13).

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