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A Note on Solvability of a Compact Perturbation of Some Linear Fredholm Operators

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In this paper, we show some conditions for a compact perturbation of a linear Fredholm operator to have a solution.

1. Preliminaries

Definition 1. Let X, Y be Banach spaces and $A: X\rightarrow Y$ a linear map. A is called *a Fredholm* [illegible] *of index* p if

a) Ker A has finite dimension d

b) Range A is closed with codimension $d^*=d-p$.

Lemma 1. *Let M be a closed subspace of a topological vector space X. If X is locally convex and has finite dimension, then M is complemented in X, i.e. there exists a closed subspace N of X such that $X=M\oplus N$.*

Proof. See [3].

Hence if A is a continuous linear Fredholm operator $X\rightarrow Y$, then $X=KerA\oplus X_2$ for some closed subspace of X by lemma 1.

Let M and N be two C^∞-manifolds of dimension n and m respectively and $f: M^n\rightarrow N^m$ be in $C^1\cap C^{n-m+1}$. We define the *rank* of f at $p\in M$ to be the rank of the $m\times n$ matrix $(\partial(y_i\circ f)/\partial x_j)$, where (x, U) and (y, V) are coordinate systems around p and $f(p)$ respectively. The point p is called a *critical point* of f if the rank of f at p is less than m (the dimension of N); if p is not a critical point of f, it is called a *regular point* of f. If p is a critical point of f, the value $f(p)$ is called a *critical value* of f. Other points in N are *regular values*; thus $q\in N$ is a regular value if and only if p is a regular point of f for every $p\in f^{-1}(q)$.

Definition 2. Let X_0 and Y be C^∞-manifolds of dimension n and $X\subset X_0$ be open with compact closure. Assume $\phi: \bar{X}\rightarrow Y$ is continuous and C^1 in X. If $y_0\in Y-\phi(\partial x)$ is a regular value of ϕ and $\phi^{-1}(y_0)=\{x_1, \ldots, x_k\}$, define $d(\phi, y_0, X)=\sum_{j=1}^{k} sgn|J\phi(x_j)|$, the *Brouwer degree* for ϕ.

Suppose now that ϕ is known to be a C^1-mapping of $\bar{\Omega}\rightarrow R^N$. Then by the Sard's theorem, we can find a sequence of regular points $\{y_n\}$ such that $y_n\rightarrow y_0$. We then define the Brouwer degree as

$$d(\phi, y_0, \Omega)=\lim_{n\rightarrow\infty} d(\phi, y_n, \Omega).$$

Furthermore, if ϕ is only known to be continous in $\bar{\Omega}$, there is a sequence of C^1-mappings $\phi_n\rightarrow\phi$ uniformly on $\bar{\Omega}$, and we set

$$d(\phi, y_0, \Omega)=\lim_{n\rightarrow\infty} d(\varphi_n, y_0, \Omega).$$

Definition 3. (a) Let D be a bounded domain in a Banach space X. If the compact mapping

$C: D \to X$ has finite dimensional range (*i.e.* $C(D) \subset X_n$, a finite dimensional subspace of X), we define the *Leray-Schauder degree* of $I+C$ at p relative to D by

$$deg\ (I+C, D, p) = d(I+C, D \cap X_n, p)$$

if the right hand side is defined.

(b) For a general compact map $C: D \to X$, approximate C by a sequence of compact mappings with finite dimensional range $C_n: D \to X_n$ such that $\sup_{x \in D} \|C_n x - Cx\| \leq 1/n$. Then define

$$deg\ (I+C,\ D, p) = \lim_{n \to \infty} d(I+C_n, D, p)$$

by (a).

Definition 4. Let S be a closed subset of a Banach space X. Suppose f is a fixed continuous mapping of $X \to Y$. Then g_0 and g_1 are *compactly homotopic* on S if there is a continuous compact mapping $h(x, t): S \times [0, 1] \to Y$ with

$$g_0(x) = f(x) + h(x, 0) \text{ and}$$
$$g_1(x) = f(x) + h(x, 1) \text{ and such that}$$
$$g(x, t) = f(x) + h(x, t) \neq 0 \text{ on } S \times [0, 1].$$

Lemma 2. *If y_0 is a regular value of ϕ, then $d(\phi, y_0, X) = deg\ (\phi, X, y_0)$.*

Proof: See [2].

Lemma 3. *Suppose D is a bounded domain in a Banach space X and $f-p \in C_l^0(\partial D, X) = \{g \mid g = I+C,\ g \neq 0$ on ∂D, $C: \partial D \to X$ is continuous and compact$\}$. Then*

(*a*) (*homotopy invariance*)

If $(h(x, t)-p) \in C_l^0(\partial D, X)$ for $t \in [0, 1]$ is a compact homotopy with $h(x, 0) = f$, then $deg(f, p, D) = deg\ (h(x, t), p, D)$.

(*b*) (*Cartesian product formula*)

If $X = X_1 \oplus X_2$ with $D_i \subset X_i$, $f = (f_1, f_2)$ with $f_i: D_i \to X_i$, $D = D_1 \times D_2$ and $p = (p_1, p_2)$, then $deg\ (f, p, D) = deg\ (f_1, p_1, D_1)\ deg\ (f_2, p_2, D_2)$ provided the right-hand side is defined.

Proof: See [1].

2. Solvability of a compact perturbation of Fredholm operators

Proposition 1. *Let X, Y be real Banach spaces and $A: X \to Y$ a bounded linear map and Fredholm. Let $K: X \to Y$ be a (nonlinear) compact map such that*

(*a*) *$K(X) \subset Range\ A$*

(*b*) *K is uniformly bounded.*

Then $Ax + K(x) = 0$ has a solution.

Proof: Since A is a Fredholm operator, A can be written

$$A = Ker A \oplus X_2$$

by lemma 1. If we restrict A on X_2, then clearly A^{-1} is linear.

Let $y_n \to y$ in $R(A)$ and $A^{-1} y_n = x_n \to z$. Since A is continuous, $Ax_n = y_n \to Az$. Therefore $y = Az$, *i.e.* $z = A^{-1} y$. Hence $A: X_2 \to R(A)$ is an isomorphism with a bounded inverse A^{-1} by the closed graph theorem. We shall prove that there exists $x_2 \in X_2$ such that $Ax_2 + K(x_2) = 0$ ....(*)

Let $x_2 \in X_2$ and set $Ax_2 = z \in R(A)$ and write $x_2 = A^{-1} z$. Then(*) takes the form

$$z + K(A^{-1} z) = 0.$$

Let $T(z)=K(A^{-1}z)$. For R sufficiently large, $(I+T)(z)=0$ has no solution for $\|z\|=R$, since K is uniformly bounded. Hence $deg(I+T,\ \|z\|\leqq R, 0)$ is defined. By the homotopy invariance in lemma 3, we see that for $0\leq t\leq 1$,

$$deg(I+T,\ \|z\|\leqq R, 0)=deg(I+tT,\ \|z\|\leqq R, 0)=1.$$

Hence $z+T(z)=0$ has a solution, *i.e.* $Ax_2+K(x_2)=0$ has a solution.

Lemma 4. *Suppose X is a topological vector space, Y is an n-dimensinonal subspace of X. Then every isomorphism of C^n onto Y is a homeomorphism.*

Proof: See [3].

Let X, Y be Banach spaces and $A: X\to Y$ a continuous linear map which is Fredholm of index 0. Decompose $X=(X_1=Ker A)\oplus X_2$, $Y=(Y_1=R(A))\oplus Y_2=QY\oplus(I-Q)Y$, where $Q: Y\to Y_1$, a projection.

Proposition 2. *Let $K: X\to Y$ be a uniformly bounded (nonlinear) compact map for which there exist positive constants R_0, ε such that*

(a) $(I-Q)K(x_1+x_2)\neq 0$ for $x_1\in X_1$, $x_2\in X_2$ and $\|x_1\|\geq R_0$, $\|x_2\|\leq\varepsilon\|x_1\|$.

(b) Leray-Schauder degree of the map $(I-Q)K(x_1)$ for $\|x_1\|=R_0$ into $Y_2-\{0\}$ at the origin is not zero. Then $Ax+K(x)=0$ has a solution.

Proof: Applying Q and $(I-Q)$ to the equation $Ax+K(x)=0$, we see that it is equivalent to the system

$$Ax_2+QK(x_1+x_2)=0$$
$$(I-Q)K(x_1+x_2)=0$$

Writing $z=Ax_2$ we obtain as in the proof of proposition 1,

$$z+QK(x_1+A^{-1}z)=0$$
$$(I-Q)K(x_1+A^{-1}z)=0$$

Since A is of index zero, X_1 and $(I-Q)Y=Y_2$ have the same dimension d. Therefore there is a linear isomorphism $B: X_1\to Y_2$ which is a homeomorphism by lemma 4. Hence setting $Bx_1=y_2$, we rewrite the system as

$$(*)\qquad z+QK(B^{-1}y_2+A^{-1}z)=0$$
$$(I-Q)K(B^{-1}y_2+A^{-1}z)=0$$

Note that the left-hand sides of these equations may be viewed as an operator of the form $I+C$, C:compact, mapping $y=z+y_2\in Y$ into Y. Here

$$C(z+y_2)=K(B^{-1}y_2+A^{-1}z)-y_2.$$

We claim that the degree of the map in a large ball $\|y\|\leqslant R$ is defined. For suppose (*) has a solution in a large ball $\|y\|=R$. Then

$$\|z\|=\|QK(B^{-1}y_2+A^{-1}z)\|\leqslant M$$

for some constant M by the uniform boundedness of K.

If we take y_2 so that $\|y_2\|$ be large enough so as to be $\|A^{-1}\|M<\varepsilon\|B\|^{-1}\|y_2\|$ which is possible if R is large, we have

$$\|A^{-1}z\|\leqslant\|A^{-1}\|M<\varepsilon\|B\|^{-1}\ \|y_2\|\leq\varepsilon\|B^{-1}y_2\|.$$

Then we must have $(I-Q)K(x_1+x_2)=(I-Q)K(B^{-1}y_2+A^{-1}z)\neq 0$, by hypothesis (a). This is a contradiction. Hence (*) cannot have a solution y on $\|y\|=R$, R large. Similarly for $0\leqslant t\leqslant 1$,

$$F_t(y): z+tQK\ (B^{-1}y_2+A^{-1}z)$$
$$(I-Q)K(B^{-1}y_2+tA^{-1}z),$$

we have $F_t(y)\neq 0$ for $\|y\|=R$ large. By the homotopy invariance $deg\ (F_t, \|y\|\leqslant R, 0)$ is independent of t. But the map F_0 is simply

$$z+y_2 \mapsto z+(I-Q)K(B^{-1}y_2).$$

This is a product map. Hence by lemma 3,

$$deg(F_0, \|y\|\leqslant R, 0)=deg\ [(I-Q)K(B^{-1}y_2),\ \|y\|\leqslant R, 0].$$

Since B is an isomorphism,

$$deg[(I-Q)K(B^{-1}y_2),\ \|y\|\leqslant R, 0]=\pm deg\ [(I-Q)K(x_1),\ \|x_1\|\leqslant R_0, 0]\neq 0$$

by (b). Therefore (*) hence $Ax+K(x)=0$ has a solution.

References

1. M. Berger, *Nonlinearity and Functional Analysis*, Academic press, 1977.
2. L. Nirenberg, *Topics in Nonlinear Functional Analysis*, Courant Institute of Mathematical Sciences, 1973~4.
3. W. Rudin, *Functional Analysis*, McGraw-Hill, 1974.