韓國數學敎育學會誌<數學敎育>
1983. 4. Vol. XXI No. 2

Journal of the Korea Society of Mathematical Education
Apr. 1983. Vol. XXI No. 2

A Note on the Prime Spectrum of a Commutative Ring

By Choi Hyo Il

1. Introduction

Let X be the set of all prime ideals of a commutative ring A with identity. For each subset of A, we denote by $V(E)$ the set of all prime ideals of A which contain E. It is well known that the sets $V(E)$ satisfy the axioms for closed set in a topological space. The set X of prime ideals of A, with the topology (Zariski) whose closed sets are the sets $V(E)$, where E runs through $\mathcal{P}(A)$, is called the ***prime spetrum of*** A and denoted by $\mathrm{Spec}(A)$. For each $f\in A$, we denote by X_f the set of prime ideals of A not containing f. Then $X_f=X-V(f)$ is therefore an open set. Let $\phi:A\to B$ be a ring homomorphism. Then for $\mathfrak{q}\in\mathrm{Spec}(B)$, $\phi^{-1}(\mathfrak{q})$ is a prime ideal of A, that is, $\phi^{-1}(\mathfrak{q})\in\mathrm{Spec}(A)$. Hence ϕ defines a mapping ϕ^* from $\mathrm{Spec}(B)$ into $\mathrm{Spec}(A)$ by $\mathfrak{q}\to\phi^{-1}(\mathfrak{q})$.

The purpose of this note is to investigate some relations between a ring homomorphism $\phi:A\to B$ and a induced map ϕ^*: $\mathrm{Spec}(B)\to\mathrm{Spec}(A)$.

The terminologies and notations used in this note will be found in [3], [4]. In particular, we will denote by $\mathfrak{N}(A)$ the set of all nilpotent elements of A (*nilradical* of A). In this note, we will assume that all rings are commutative ring with identity.

2. Main Results

Through this section, $\phi:A\to B$ is a ring homomorphism, $X=\mathrm{Spec}(A)$, $Y=\mathrm{Spec}(B)$ and $\phi^*:Y\to X$ is a induced map.

Lemma 2.1 *For each $f\in A$, $X_f=\phi$ if and only if f is a nilpotent element of A.*

proof. Assume that f is nilpotent, then $f^n=0$ for some $n>0$. Thus $f\in\mathfrak{p}$ for all $\mathfrak{p}\in X$, which implies $V(f)=X$. That is, $X_f=\phi$.

Conversely, suppose that X_f is empty. Then $V(f)=X$. Hence $f\in\mathfrak{p}$ for all $\mathfrak{p}\in X$. Since the nilradical $\mathfrak{N}(A)$ of A is the intersection of all prime ideals of A, we see that f is a nilpotent element.

Theorem 2.2 *For $f\in A$, $\phi^{*-1}(X_f)=Y_{\phi(f)}$ and hene ϕ^* is continuous.*

proof. A prime ideals $\mathfrak{p}$ ia a member of ϕ^{*-1} (X_f) if and only if $\phi^*(\mathfrak{p})\in X_f$. By definition of X_f, $\phi^*(\mathfrak{p})\in X_f$ if and only if $f\notin\phi^*(\mathfrak{p})$. Since $\phi^*(\mathfrak{p})=\phi^{-1}(\mathfrak{p})$, $f\notin\phi^*(\mathfrak{p})$ is equivalent to $f\notin\phi^{-1}(\mathfrak{p})$, that is, $\phi(f)\notin\mathfrak{p}$. Moreover, this is equivalent to $\mathfrak{p}\in Y_{\phi(f)}$. This completes the proof.

Theorem 2.3 *If ϕ is surjective, then ϕ^* is a homeomorphism of Y onto the closed subset $V(Ker(\phi))$ of X.*

proof. By theorem 2.2, ϕ^* is continuous. We have to show that ϕ^* is injective, surjective and that ϕ^{*-1} is continuous.

i) φ^* is injective: For $\mathfrak{p}$, $\mathfrak{q} \in Y$, $\varphi^*(\mathfrak{p}) = \varphi^*(\mathfrak{q})$ implies that

$\varphi^{-1}(\mathfrak{p}) = \varphi^{-1}(\mathfrak{q})$. Thus we have $\mathfrak{p} = \mathfrak{q}$.

ii) φ^* is surjective: For each $\mathfrak{p} \in V(\mathrm{Ker}(\varphi))$, we must find $\mathfrak{q} \in Y$ such that $\varphi^*(\mathfrak{q}) = \mathfrak{p}$. Since φ is surjective, we have $A/\ker(\varphi) \cong B$ by lst Isomorphism theorem. Moreover, $\mathfrak{p}/\ker(\varphi)$ is a prime ideal of $A/\ker(\varphi)$. Thus there exists an ideal $\mathfrak{q}$ of B, which corresponds to $\mathfrak{p}/\ker(\varphi)$. Then $\varphi^*(\mathfrak{q}) = \varphi^{-1}(\mathfrak{q}) = \mathfrak{p} + \ker(\varphi) = \mathfrak{p}$.

iii) ϕ^{*-1} is continuos: Since φ^* is bijective, φ^{*-1} is well defined. By Treorem 2.2, $\varphi^*(Y_{\phi(f)}) = X_f$. Hence φ^* is open mapping and this means that φ_*^{-1} is continuous.

Theorem 2.4 *If φ is injective, then $\varphi^*(Y)$ is dense in X. Furthermore, $\varphi^*(Y)$ is dense in X if and only if $Ker(\varphi) \subseteq \mathfrak{N}(A)$.*

proof. If φ is injective, then $\mathrm{Ker}(\phi) \subseteq \mathfrak{N}(A)$. Therefore, it suffices to show that the latter statement holds.

First assume that $\varphi^*(Y)$ is dense in X. Let $f \in \mathrm{Ker}(\varphi)$. Suppose $X_f \neq \phi$. Then $\varphi^*(Y) \cap X_f \neq \phi$. Hence there exists $\mathfrak{q}$ in Y such that $\varphi^{-1}(\mathfrak{q}) \in X_f$. That is, $\mathfrak{q} \in \varphi^{*-1}(X_f)$. By Theorem 2.2, we have $\varphi_*^{-1}(X_f) = Y_{\phi(f)} = \phi$. Thus $\mathfrak{q} \in \phi$, a contradiction. Hence $X_f = \phi$. By Lemma 2.1, we have $f \in \mathfrak{N}(A)$. That is, $\mathrm{Ker}(\varphi) \subseteq \mathfrak{N}(A)$.

Conversely, suppose $\mathrm{Ker}(\varphi) \subseteq \mathfrak{N}(A)$. Assume that $X_f \neq \phi$. Then f is not a nilpotent element. Hence $\varphi(f)$ is not nilpotent. Thus by Lemma 2.1, we have $Y_{\phi(f)} \neq \phi$. Moreover, $\varphi^{*-1}(X_f) = Y_{\phi(f)}$ by Theorem 2.2. This implies that there exists $\mathfrak{q} \in \varphi_*^{-1}(X_f)$. That is, $\varphi^*(\mathfrak{q}) \in X_f$. Therefore, $X_f \cap \varphi^*(Y) \neq \phi$ and hence $\varphi^*(Y)$ is dense in X.

References

[1] Hideyuki Matsumura, *Commutative Algebra*, The Benjamin Pub. Co. (1980)

[2] James Dugundji, *Topology*, Allyn & Bacon, Inc. (1968)

[3] M.F. Atiyah and I.G. MacDonald, *Introduction to Commutative Algebra*, Addison-Wesley Publishing Co., Inc. (1969)

[4] Nicolas Bourbaki, *Commutative Algehra*, Addison-Wesley Publishing Co., Inc. (1972)