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Acyclic Orientations of Matroids

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1. Introduction

The origin of matroids is the paper of Whitney[3] where we may see how a finite subset of a vector space over a fields yields a matroid. However matroids do not capture certain sign properties of vector spaces over *ordered* field. Recently, the properties of positive dependence relation[2] showed the structure of an oriented matroid.

The present paper is devoted to proving that an extreme point P in an acyclic oriented matroid has a signed cocircuit Y such that $Y^+=P$.

2. Basic Concepts and Notations

A *signed set* X is a set $\underline{X}$, called the *set underlying* X, and a mapping $sg_X: \underline{X}\to\{-1,1\}$, called the *signature* of X. A collection of two sets $X^+=\{x\in\underline{X}: sg_X(x)=1\}$ and $X^-=\{x\in\underline{X}: sg_X(x)=-1\}$ is a partition of $\underline{X}$. The *opposite* of X, denoted $-X$, is the signed set having $(-X)^+=X^-$ and $(-X)^-=X^+$. If $\underline{X}$ is a subset of some set E, then X will be called a *signed subset of E*.

An *oriented matroid* $\hat{M}$ on a finite set E is defined by an ordered pair (E,Θ) where Θ is a collection of non-empty signed subsets of E, called *signed circuits*, such that:

(1) If X_1, $X_2\in\Theta$, $\underline{X}_1\subset\underline{X}_2$, then $X_1=X_2$ or $X_1=-X_2$.

(2) If $X\in\Theta$, then $X\neq\phi$ and $-X\in\Theta$

(3) (Signed elimination property)

If X_1, $X_2\in\Theta$, $x\in(X_1^+\cap X_2^-)\cup(X_1^-\cap X_2^+)$, then there exists $X_3\in\Theta$ such that $X_3^+\subset(X_1^+\cup X_2^+)\backslash x$ and $X_3^-\subset(X_1^-\cup X_2^-)\backslash x$.

If $M=(E,\Theta)$ is an oriented matroid, the above condition (3) is equivalent to the following condition:

(3′) If X_1, $X_2\in\Theta$, $x\in(X_1^+\cap X_2^-)\cup(X_1^-\cap X_2^+)$ and $y\in(X_1^+\backslash X_2^-)\cup(X_1^-\backslash X_2^+)$ then there exists $X_3\in\Theta$ such that $X_3^+\subset(X_1^+\cup X_2^+)\backslash x$, $X_3^-\subset(X_1^-\cup X_2^-)\backslash x$ and $y\in X_3$. [1]

For a collection Θ of signed sets, let $\underline{\Theta}=\{\underline{X}: X\in\Theta\}$. If $\hat{M}=(E,\Theta)$ is an oriented matroid, then $\underline{\hat{M}}=(E,\underline{\Theta})$ is a matroid. [3]

Conversly, let M be a matroid on E with the circuits $\mathcal{C}$, and $\hat{M}$ be an oriented matroid on E with the signed circuits Θ. If $\underline{\Theta}=\mathcal{C}$, $\Theta=-\Theta=\{-X: X\in\Theta\}$, then Θ is called a *circuit signature* (or *orientation*) of M. Let $M^*=(E,\mathcal{C}^*)$ be the dual matroid of M where $\mathcal{C}^*$ is the set of cocircuits of M. Accordingly, a cocircuit signature Θ^* of M is a circuit signature of M^*.

We have the relation between Θ and Θ^* such that;

(4) (Orthogonality property)

If $X \in \Theta$ and $Y \in \Theta$ such that $\underline{X} \cap \underline{Y} \neq \phi$, then $(X^+ \cap Y^+) \cup (X^- \cap Y^-) \neq \phi$ and $(X^+ \cap Y^-) \cup (X^- \cap Y^+) \neq \phi$.

Let $\hat{M}=(E, \Theta)$ be an oriented matroid. We define $\hat{M}$ to be *acyclic* if $\hat{M}$ contains no positive circuits (signed circuits X with $X^-=\phi$).

Let $M=(E, \mathcal{C})$ be a matroid on E with the circuits $\mathcal{C}$. A *point* of M is a closed subset of E of rank 1. A *hyperplane* of M is a maximal proper closed subset of E. We link hyperplanes and cocircuits;

A set H is a hyperplane of the matroid $M=(E, \mathcal{C})$ if and only if $E \backslash H$ is a cocircuit of M.[3]

Let $\hat{M}=(E, \Theta)$ be an acyclic oriented matroid. We call Y^+ an *open half space* of $\hat{M}$ where Y is a signed cocircuit of $\hat{M}$. A hyperplane H of $\hat{M}=(E, \Theta)$ is called a *facet* of $\hat{M}$ if there is a positive cocircuit Y ($\underline{Y}=Y^+$) such that $E \backslash H=\underline{Y}=Y^+$. And any intersections of facets is called a *face* of $\hat{M}$. In particular, an *extreme point* of $\hat{M}$ is a face of rank 1.

Let $\hat{M}=(E, \Theta)$ be an oriented matroid on E with the signed circuits Θ, and let $e \in E$. The set $\Theta \backslash e$ obtained from Θ by deleting e in Θ is defined by $\Theta \backslash e=\{X \in \Theta : e \notin X\}$. The set Θ / e obtained from Θ by contracting e in Θ is defined by $\Theta / e = Min\{X \backslash e : X \in \Theta, X \backslash e = \phi\}$, where $X \backslash e$ denotes the signed set Z such that $Z^+=X^+ \backslash e$, $Z^-=X^- \backslash e$, and $Min(\Theta)=\{X \in \Theta : X' \in \Theta$ and $\underline{X}' \subset \underline{X}$ imply $\underline{X}'=\underline{X}\}$.

3. Main Theorem

Lemma 1. *Let $\hat{M}=(E, \Theta)$ be an oriented matroid on a finite set E with a circuit signature Θ, and let $e \in E$. $(E, \Theta \backslash e)$ and $(E, \Theta / e)$ are oriented matroids.*

Proof is omitted [1].

Oriented matroids $\hat{M} \backslash e$ and $\hat{M}/e$ denote oriented matroids $(E, \Theta \backslash e)$ and $(E, \Theta / e)$ respectively.

Lemma 2. *Let $\hat{M}$ be an acyclic oriented matroid on E. Every face F of $\hat{M}$ is a closed subset of E such that $\hat{M}/F$ is acyilc.*

Proof. Since F is an intersection of facets of $\hat{M}$. $E \backslash F$ is an union of positive cocircuits of $\hat{M}$. By Lemma 1, $E \backslash F$ is an union of positive cocircuits of $\hat{M}/F$. Therefore $\hat{M}/F$ is acyclic. Obivously, F is a closed set.

Lemma 3. *Let $\hat{M}=(E, \Theta)$ be an oriented matroid, $X_1, X_2, \ldots, X_n$ be positive circuits, and X be a signed circuit. Then for any $e \in X \backslash \bigcup_{i=1}^{n} X_i$ there is a signed circuit Z of $\hat{M}$ such that $e \in Z$, $Z^+ \subset X^+ \cup (\bigcup_{i=1}^{n} X_i)$, $Z^- \subset X^- \backslash (\bigcup_{i=1}^{n} X_i)$.*

Proof. Suppose that Z is a signed circuit of $\hat{M}$ such that $e \in Z \subset X \cup (\bigcup_{i=1}^{n} X_i)$, $Z^+ \subset X^+ \cup (\bigcup_{i=1}^{n} X_i)$ and $Z^- \subset X^- \cup (\bigcup_{i=1}^{n} X_i)$, and Z is chosen such that $|Z^- \cap (\bigcup_{i=1}^{n} X_i)|$ is minimal.

Let $x \in Z^- \cap X_1^+$. By the definition of an oriented matroid, there is a signed circuit W of $\hat{M}$ such that $W^+\subset Z^+\cup X_1^+\backslash x$, $W^-\subset Z^-\backslash x$. Hence $e\in W$. $W^+\subset X^+\cup(\bigcup_{i=1}^{n}X_i)$, $W^-\subset X^-\cup(\bigcup_{i=1}^{n}X_i)$ and $|W^-\cap(\bigcup_{i=1}^{n}X_i)|<|Z^-\cap(\bigcup_{i=1}^{n}X_i)|$. This contradicts the minimality of $|Z^-\cap(\bigcup_{i=1}^{n}X_i)|$.

Lemma 4. *Let $\hat{M}=(E,\Theta)$ be an acyclic orinted matroid, P be a point of $\hat{M}$. If $\hat{M}/P$ is not acyclic, then there is a signed circuit X of $\hat{M}$ such that for an element e of E, $X^-=\{e\}$ and $e\in P$.*

Proof. Suppose that $\hat{M}/P$ is not acyclic. Then there is a positive circuit X' of $\hat{M}/P$. Let X be a signed circuit of $\hat{M}$ such that $X'=X\backslash P$. Hence $X^-\subset P$. On the other hand, we have $|X\cap P|\leq 1$. Therefore since $\hat{M}$ is acyclic, there is an element e of E such that $X^-=\{e\}$, $e\in P$.

Theorem 5. *Let $\hat{M}=(E,\Theta)$ be an acyclic oriented matroid on E with the signed circuits Θ. A point P of $\hat{M}$ is an extreme point if and only if there is a signed cocircuit of $\hat{M}$ such that $Y^+=P$.*

Proof. If P is an extreme point of $\hat{M}$, then $E\backslash P$ is an union of positive cocircuits. Since $\hat{M}$ is acyclic, $\hat{M}$ has no loops. Hence there is a signed cocircuit Y' such that $P\subset Y'^+$. By Lemma 3, we have a signed cocircuit Z such that $Z^-\subset P$. Since $\hat{M}$ is acyclic, $Z^-=P$. By definition of $\hat{M}^*$ there is a signed cocircuit Y of $\hat{M}$ such that $Y^+=P$. .

Conversely, suppose that there is a signed cocircuit Y of $\hat{M}$ such that $Y^+=P$. We show that P is a face of $\hat{M}$. By Lemma 2, it suffices to prove that $\hat{M}/P$ is acyclic. Suppose that $\hat{M}/P$ is not acyclic. By Lemma 4, there is a signed circuit X of $\hat{M}$ such that for an $e\in E$, $X^-=\{e\}$ and $e\in P$; so $X^-=\{e\}$ and $X^-\cap Y^+=\{e\}$. Therefore $(X^+\cap Y^+)\cup(X^-\cap Y^-)=\phi$. This contradicts the orthogonal property (4).

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