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A Necessary Condition for the Extension of the Erdos-Rényi Law of Large Numbers

By Choi, Yong Kab
Gyeongsang National University, Korea

For a sequence $X_1, X_2, \cdots$ of indepent, identically distributed random variables (i.i.d. r.v.s) with $S_n = X_1 + \cdots + X_n$, Kolmogorov-Marcinkiewicz Strong Law of Large Numbers [1, 2] asserts that if $E(|X_1|^r) < \infty$, then $n^{-1/r}(S_n - {}_nC_r) \xrightarrow{\text{a.s.}} 0$ for $0 < r < 2$, where $C_r = 0$ if $0 < r < 1$ and $C_r = E(X_1)$ if $1 \leq r < 2$. From this point of view, for the above sequence $X_1, X_2, \cdots$ with m.g.f. $f(t) < \infty$ for all $t > 0$, Y.K. Choi[3] studied the maximum $D_r(N, K)$ of the $N-K+1$ averages of the form $K^{-1/r}$ $(S_{n+k} - S_n)$ for $0 \leq n \leq N-K$, where $S_0 = 0$ and $0 < r \leq 1$, who proved that, for a wide range of positive numbers a, $\lim_{N\to\infty} D_r(N, [(C(a)\log N)^{(2-r)/r}]) = a$ with probability 1 (w.p.1), where $C(a)$ is a known constant depending on a and the distribution of X_1. This is an extension of the Erdös-Rényi new law of large numbers. At first, J. Steinebach (1978) proved that the existence of m.g.f. $f(t)$ is a necessary condition for the Erdös-Rényi law of large numbers in the case $r = 1$. The purpose of this paper is to show that $\limsup_{N\to\infty} D_r(N, [(C(a)\log N)^{(2-r)/r}]) = \infty$ for $0 < r \leq 1$, if the m.g.f. does not exist for all $t > 0$.

For $r = 1$, P. Erdös and A. Rényi (1970) developed their original "A new law of large numbers" as follows.

Theorem 1. (Erdös-Rényi) Let $X_1, X_2, \cdots$ *be a sequence of nondegenerate i.i.d. r.v.s on a probability space* $(\Omega, \mathcal{A}, P)$ *with m.g.f.* $f(t) < \infty$ *for* $t \in [0, t_1)$, $0 < t_1 < \infty$. *For a positive number a and for a known constant* $C(a)$ *depending on a and the distribution of* X_1, *let* $\inf_t f(t)\exp(-ta) = \exp(-1/C(a))$. *Then* $C(a) > 0$ *and*

(1) $\lim_{N\to\infty} D_1(N, [C(a)\log N]) = a \quad w.p.1,$

where $[x]$ *denotes the integral part of* x.

Proof. See [4].

The following Theorem 2 states the extension of Erdös-Rényi law of large numbers for $0 < r \leq 1$.

Theorem 2. [3] *Let* $X_1, X_2, \cdots$ *be a sequence of nondegenerate i.i.d. r.v.s on* $(\Omega, \mathcal{A}, P)$ *with* $f(t) < \infty$ *for* $t \in [0, t_1)$ *and* $0 < r \leq 1$. *Then for every* $a > 0$ *and* $C(a) > 0$ *such that* $\inf_t f(t)\exp(-ta) = \exp(-1/C(a))$, *we have*

(2) $\lim_{N\to\infty} D_r(N, [(C(a)\log N)^{(2-r)/r}]) = a \quad w.p.1.$

Since the existence of m.g.f. yields an exponential convergence rate for the large deviation probabilities $P(n^{-1}S_n \geq a)$, the existence of m.g.f. is sufficient for proving (1) and (2). But, it is a question whether the existence of m.g.f. is also necessary to retain assertions (1) and (2) by exponential large deviation probabilities. From Petrov and Sirokova (1973) we get a positive answer as follows.

Theorem 3. [5] *Let* $X_1, X_2, \cdots$ *be a sequence of i.i.d. r.v.s with*

$$P(K_N^{-1/r}S_{K_N} \geq a) \leq A\rho^{K_N},\ K_N=1,2,\cdots;\ 0<r\leq 1,$$

for some constants a, A and $0<\rho<1$. *Then there exists some* $t_1>0$ *such that* $f(t)<\infty$ *for* $t\in[0,t_1$

Proof. See [5], and it follows from the fact that

$$P(K_N^{-1/r}S_{K_N}\geq a)\leq P(K_N^{-1}S_{K_N}\geq a) \text{ for } 0<r\leq 1.$$

Corollary 4. *Let* $X_1, X_2, \cdots$ *be a sequence of i.i.d. r.v.s with* $f(t)=\infty$ *for all* $t>0$. *Then for constants a and* ρ $(0<\rho<1)$,

$$\text{limsup}_{N\to\infty}P(K_N^{-1/r}S_{K_N}\geq a)/\rho^{K_N}=\infty.$$

This Corollary 4 is essential to prove the following Theorem 5 for $r=1$.

Theorem 5. [6] *Let* $X_1, X_2, \cdots$ *be a sequence of i.i.d. r.v.s with* $f(t)=\infty$ *for all* $t>0$. *Then have*

$$\text{limsup}_{N\to\infty}D_1(N,[C(a)\log N])=\infty \quad w.p.1$$

for every positive constant $C(a)$.

Now we are ready to state and prove the main Theorem 6.

Theorem 6. *Let* $X_1, X_2, \cdots$ *be a sequence of i.i.d. r.v.s with* $f(t)=\infty$ *for all* $t>0$. *Then we ha*

$$\text{limsup}_{N\to\infty}D_r(N,[(C(a)\log N)^{1/q}])=\infty \quad w.p.1$$

for every positive constant $C(a)$ *depending on a and the distribution of* X_1, *where* $q=r/(2-r)$ *a* $0<r\leq 1$.

Proof. For arbitrary a and $0<\rho<1$, Corollary 5 implies the existence of a subsequence $\{K_{N_j}\}$ $_{1,2,\cdots}$ of integers such that

$$P(K_{Nj}^{-1/r}S_{K_{Nj}}\geq a)\geq\sigma^{K_{Nj}},\ j=1,2,\cdots.$$

Let $K_N=[(C(a)\log N)^{1/q}]$, then

$$\begin{aligned}
&P(D_r(Nj,\ K_{Nj})<a)\\
&\leq P[\max_{i=1,\cdots,[Nj/K_{Nj}]}\{(S_{iK_{Nj}}-S_{(i-1)K_{Nj}})K_{Nj}^{-1/r}<a\}]\\
&\leq\{1-P(S_{K_{Nj}}\cdot K_{Nj}^{-1/r}\geq a)\}^{[Nj/K_{Nj}]}\\
&\leq\{1-\rho^{K_{Nj}}\}^{[Nj/K_{Nj}]}\leq\exp(-\rho^{K_{Nj}}[Nj/K_{Nj}]).
\end{aligned}$$

If $1>\rho=\exp(-1/C_1(a))$, where $C_1(a)>C(a)$, we have

$$\begin{aligned}
&\rho^{K_{Nj}q}\geq\rho^{C(a)\log Nj}=Nj^{-C(a)/C_1(a)}=Nj^{-(1-2\delta)}\\
&\quad\geq\rho^{(C(a)\log Nj)^{1/q}},
\end{aligned}$$

and

$$\rho^{K_{Nj}q}\geq\rho^{K_{Nj}}\geq\rho^{(C(a)\log Nj)^{1\setminus q}}.$$

Hence we can take a suitable $\delta>0$ such that

(3) $\rho^{K_{Nj}}\geq Nj^{-(1-2\delta)}$.

For all sufficiently large j, say $j\geq j_0$,

(4) $[Nj/K_{Nj}]=[Nj/[(C(a)\log Nj)^{1/q}]]\geq Nj^{1-\delta}$,

because there is Nj_0 such that $C(a)\cdot\log Nj\leq Nj^{q\delta}$ for all $Nj\geq Nj_0$, if $q\delta\not\to 0(0<q\delta<1/2)$.

From (3) and (4), we have

$$P(D_r(Nj,\ K_{Nj})<a)\leq \exp(-Nj^{\delta}) \text{ for all } j\geq j_0.$$

By the integral test,

$$\sum_{j=j_0}^{\infty} P(D_r(Nj,\ K_{Nj})<a)<\infty.$$

Thus, by the Borel-Cantelli lemma,

$$\liminf_{N\to\infty} D_r(Nj,\ K_{Nj})\geq a \quad \text{w.p.1.}$$

Therefore, we have

$$\limsup_{N\to\infty} D_r(N,\ K_N)\geq a \quad \text{w.p.1.}$$

Since a is arbitrary, the proof is complete.

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