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A Note on π-Mapping of Metric Spaces

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1. Introduction

We know that one of the fundamental directions in general topolgy is to determinate connections induced between classes of spaces by means of mappings of different types. The metrizable spaces are a fundamental class of topolgical spaces; thus one of the problems is the description in intrinsic terms of the spaces of the partitionings of metric spaces under various restrictions on the elements of the partitioning. Before V.I. Ponomarev have explained a connection between a class of metric spaces and a class of spaces having a refining sequence of coverings. And also, he have defined π-mapping, i.e. the mapping $f: X\to Y$ of the metric space Y is called a *π-mapping*, if for any point $x\in Y$ and any neighborhood of it Nx, $d(f^{-1}x,\ X\setminus f^{-1}Nx)>0$.

In this note we will give the other definition based on π-mapping and prove the one of theorems in connection with it.

2. Definition and Main Theorem

Definition. If the set S is closed in X iff for every point $x\in X\setminus S$ there exists a natural number such that $(\bigcap_{i=1}^{n}\nu_i x)\cap S=\phi$, where $\nu_i x=U\{H_\alpha\in\nu_i\,|\,x\in H_\alpha\}$, the system of coverings (may not be open) $\{\nu_n=\{H_\alpha\,|\,\alpha\in A_n\}\,;\ n=1,2,\cdots\}$ of the space is said to be *semirefined*.

Theorem. *The following conditions are equivalent:*

(1) *the space X possesses a semirefined sequence of coverings.*

(2) *the space X is a factor π-image of some metric space.*

Proof. (2)$\Rightarrow$(1). Let $f: Z\to X$ be a factor π-mapping of the metric space Z onto the space X. We set $\nu_n=\{fU_\alpha\,|\,U_\alpha\subseteq Z$ and diam $U_\alpha<1/n\}$. We prove that the system $\{\nu_n\,|\,n=1,2,\cdots\}$ is semirefined. Let the set S be closed in X and let $x\in X\setminus S$. Then we have $d(f^{-1}x, f^{-1}S)>1/n>0$ from the condition. Thus we have $\nu_n x\cap S\neq\phi$. Moreover, let the set S be such that for any point $x\in X$ there can be found a natural number n such that $\nu_n x\cap S=\phi$. Next we prove that the set $f^{-1}S$ is closed in Z i.e. the set S is closed in X. In fact, if $z\in f^{-1}S$, then $f^{-1}fz\cap f^{-1}S=\phi$, and for some n we have $\nu_n fz\cap S=\phi$, and hence $N(z,1/n)\cap f^{-1}S=\phi$. This proves the semirefinement of the coverings $\{\nu_n\,|\,n=1,2,\cdots\}$.

(1)$\Rightarrow$(2). Let $\{\nu_n=\{H_\alpha\,|\,\alpha\in A_n\},\ n=1,2,\cdots\}$ have a sequence of semirefined coverings of the space X. We set $Z=\prod_{n=1}^{\infty}A_n$, where A_n is a discrete space, i.e. for any $\alpha_1,\alpha_2\in A_n$ we have $d_n(\alpha_1,\alpha_2)=1$ if $\alpha_1\neq\alpha_2$.

The metric on the space Z is given by: if $(\alpha_1,\cdots,\alpha_n,\cdots)$, $(\beta_1,\cdots,\beta_n,\cdots)\in Z$, then $d((\alpha_1,\cdots,\alpha_n,\cdots),(\beta_1,\cdots,\beta_n,\cdots))=[\sum_{n=1}^{\infty}\frac{1}{2^n}d_n^2(\alpha_n,\beta_n)]^{1/2}$. If $\bigcap_{n=1}^{\infty}H_{\alpha_n}\neq\phi$, then we will call the point $(\alpha_1,\cdots,\alpha_n,\cdots)\in Z$

labelled and denote the union of the labelled points by Z_0. Let $f: Z_0 \rightarrow X$, where $f(\alpha_1, \cdots, \alpha_n, \cdots) = \bigcap_{n=1}^{\infty} H_{\alpha_n}$.

It is evident that $\bigcap_{n=1}^{\infty} H_{\alpha_n}$ is composed of only one point. We will designate $A_n x = \{\alpha | \alpha \in A_n$ an $x \in H_\alpha\}$. It is easy to prove the continuity of the mapping f and the formula

$$f^{-1}x = \prod_{n=1}^{\infty} A_n x, \text{ for any point } x \in X \quad (*$$

holds. First we prove that f has a π-mapping. Let the point x and its neighborhood Nx be arl itrary. There exists a natural number n such that $\bigcap_{i=1}^{n} \nu_i x \subseteq Nx$. Then, if $(\alpha_1, \cdots, \alpha_n, \cdots)$, $(\beta_1, \cdots \beta_n, \cdots) \in Z_0 \backslash f^{-1}Nx$, then $\alpha_i \neq \beta_i$, where $i \leq n$, and so $d((\alpha_1, \cdots, \alpha_n, \cdots), (\beta_1, \cdots, \beta_n, \cdots)) \geq 1/n$.

Thus $d(f^{-1}x, Z_0 \backslash f^{-1}Nx) \geq 1/n > 0$. Next we prove that the mapping f is factorial. In other word: we prove that if the set S is not closed in X, then the set $f^{-1}S$ is not closed in X. In fact, if tl set S is not closed in X, then there is a point $x \in X \backslash S_n$ and a set $H_{\alpha_n} \in \nu_n$ such that $x \in H_{\alpha s}$ an $(\bigcap_{j=1}^{n} H_{\alpha_i}) \cap S \neq \phi$ for any natural number n.

It is clear that $(\alpha_1, \cdots, \alpha_n, \cdots) \in f^{-1}x$. Finally we show that $(\alpha_1, \cdots, \alpha_n, \cdots) \in [f^{-1}S]_{Z_0}$. From tl condition $(\bigcap_{i=1}^{n} H_{\alpha_i}) \cap S \neq \phi$, it follows that there exists a point $z_m = (\beta_1, \cdots, \beta_n, \cdots) \in f^{-1}S$, whei $\beta_i \neq \alpha_i$ for $i = 1, 2, \cdots, m$. So the sequence $\{z_m | m = 1, 2, \cdots\}$ converges to the point $(\alpha_1, \cdots, \alpha_n, \cdots)$ Therefore the work above shows that this theorem is proved, as desired.

Corollary. *The following conditions are equivalent:*

(1) *The space X possesses a semirefinded sequence of pointwise finite coverings.*

(2) *The space X has a factorial bicompact form of some metric space.*

Proof. From the formula(*) and above theorm it is obvious by similar reasoning.

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