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A study on almost strongly θ-continuous functions

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I. Introduction

Recently many authors studied various forms of continuity on topological spaces.

Furthermore, they have defined a function $h: X\to Y$ from a topological space X into a topological space Y to be *strongly θ-continuous* [1] (resp. *almost-continuous* [3], *θ-continuous* [3], *δ-continuous* [2]) if for each $x\in X$ and each open neighborhood V of $f(x)$, there exists an open neighborhood U of x such that $f(Cl(U))\subset V$ (resp. $f(U)\subset Int(Cl(V))$, $f(Cl(U))\subset Cl(V)$, $f(Int(Cl(U)))\subset Int(Cl(V))$).

The purpose of this paper is to introduce the concepts of almost strongly θ-continuous functions which are weaker than strongly θ-continuous functions and stronger than θ-continuous functions and δ-continuous functions, and to investigate some properties of them.

II. Basic concepts

Definition 2.1. A function $f: X\to Y$ is said to be *almost strongly θ-continuous* if for each $x\in X$ and each open neighborhood V of $f(x)$, there exists an open neighborhood U of x such that $f(Cl(U))\subset Int(Cl(V))$.

Theorem 2.2. *The following diagram implications hold:*

strongly θ-continuous
↓①
almost strongly θ-continuous
↙②　　　③↘
δ-continuous←⋯*independent*⋯→*θ-continuous*

Proof. These are obvious from definitions.

Example 2.3. The converse of ① need not be true. For let τ be the finite complement topology on R and let $f: (R,\tau)\to(R,\tau)$ be the identity function. Then, f is almost strongly θ-continuous, but it is not strongly θ-continuous. Because $Int(Cl(U))=R$ for each open set U of (R,τ).

Example 2.4. The converse of ② need not be true. For let $X=\{a,b,c\}$ and $\tau=\{X,\emptyset,\{a\},\{c\},\{a,c\}\}$ and let $f: (X,\tau)\to(X,\tau)$ be the identity function. Then, f is δ-continuous, but it is not almost strongly θ-continuous. Because $\{a,b\}=Cl(\{a\})\not\subset Int(Cl(\{a\}))=\{a\}$.

Example 2.5. The converse of ③ need not be true. For the function f in example 2.4 is θ-continuous.

In [6], A subset S of a topological space X is said to be ***regular open*** (resp. ***regular closed***) if $Int(Cl(S))=S$ (resp. $Cl(Int(S))=S$).

In [5], N.V. Velicko defined the followings:

(i) The *δ-closure* (resp. *θ-closure*) of a subset A of a topological space X, denoted by $Cl_\delta(A)$ (resp. $Cl_\theta(A)$), is $\{x\in X \mid$ every regular open neighborhood of x meets $A\}$ (resp. $\{x\in X \mid$ every closed neighborhood of x meets $A\}$). The subset A is *δ-closed* (resp. *θ-closed*) if $Cl_\delta(A)=A$ (resp. $Cl_\theta(A)=A$).

(ii) The *δ-interior* (resp. *θ-interior*) of a subset A of a topological space X, denoted by $Int_\delta(A)$ (resp. $Int_\theta(A)$), is $\{x\in X \mid$ some regular open neighborhood of x lies in $A\}$ (resp. $\{x\in X \mid$ some closed neighborhood of x lies in $A\}$). The subset A is *δ-open* (resp. *θ-open*) if $Int_\delta(A)=A$ (resp. $Int_\theta(A)=A$). Of course both δ-open sets and θ-open sets are open, and both δ-closed sets and θ-closed sets are closed. Furthermore, the complement of a δ-open (resp. θ-open) set is δ-closed (resp. θ-closed) and the complement of a δ-closed (resp. θ-closed) set is δ-open (resp. θ-open).

Theorem 2.6. *For any function $f: X\rightarrow Y$ the followings are equivalent.*

(a) *f is almost strongly θ-continuous.*

(b) *For each $x\in X$ and each regular open set V containing $f(x)$, there exists an open set U containing x such that $f(Cl(U))\subset V$.*

(c) *For every regular open set V of Y, $f^{-1}(V)$ is θ-open in X.*

(d) *For every regular closed set F of Y, $f^{-1}(F)$ is θ-closed in X.*

(e) *The inverse image of a δ-closed set is θ-closed.*

(f) *The inverse image of a δ-open set is θ-open.*

Proof. (a)⇔(b), (c)⇔(d), (e)⇔(f): These proofs are clear from definitions.

(b)⇒(c): Let V be a regular open set of Y and let $x\in f^{-1}(V)$, then $f(x)\in V$. By (b), there exists an open set U containing x such that $f(Cl(U))\subset V$. Hence $Cl(U)\subset f^{-1}(V)$, and so $x\in Int_\theta(f^{-1}(V))$. This shows that $f^{-1}(V)\subset Int_\theta(f^{-1}(V))$. Since $f^{-1}(V)\supset Int_\theta(f^{-1}(V))$ is clear, $f^{-1}(V)=Int_\theta(f^{-1}(V))$ Therefore $f^{-1}(V)$ is θ-open.

(c)⇒(f): Let V be δ-open in Y. Then for each $y\in V$, there exists a regular open set W_y containing y such that $W_y\subset V$. Thus, $V=\bigcup_{y\in V} W_y$. This shows that $f^{-1}(V)=f^{-1}(\bigcup_{y\in V} W_y)=\bigcup_{y\in V} f^{-1}(W_y)$. But by (c), each $f^{-1}(W_y)$ is θ-open. Hence $f^{-1}(V)$ is θ-open.

(f)⇒(b): Let $x\in X$ and let V be a regular open set of Y containing $f(x)$. Then V is δ-open, and hence by (f) $f^{-1}(V)$ is a θ-open set containing x. Thus, there exists an open set U containing x such that $Cl(U)\subset f^{-1}(V)$. Therefore $f(Cl(U))\subset f(f^{-1}(V))\subset V$.

Lemma 2.7. *A space X is Hausdorff if and only if for any x_1, x_2 in X, there exist open sets U and V containing x_1, and x_2 respectively such that $Int(Cl(U))\cap Int(Cl(V))=\emptyset$.*

Proof. $U\cap V=\emptyset$ implies that $Int(Cl(U))\cap V=\emptyset$. Hence $Int(Cl(U))\cap Int(Cl(V))=\emptyset$.

Theorem 2.8.

Let $f: X\rightarrow Y$ be an injective almost strongly θ-continuous function and let Y be Hausdorff. Then X is Urysohn.

Proof. Let $x_1\neq x_2$ belong to X. Then $f(x_1)\neq f(x_2)$. Since Y is Hausdorff, by lemma 2.7 there exist disjoint regular open sets V_1 and V_2 containing $f(x_1)$ and $f(x_2)$ respectively. Thus, there exist open sets U_1 and U_2 containing x_1 and x_2 respectively such that $f(Cl(U_1))\subset V_1$ and $f(Cl(U_2))\subset V_2$ because f is almost strongly θ-continuous. It follows that $Cl(U_1)\cap Cl(U_2)=\emptyset$, from which we conclude that X is Urysohn.

Theorem 2.9. *If $f: X\to Y$ is almost strongly θ-continuous and $g: Y\to Z$ is δ-continuous, then the composition $g\circ f: X\to Z$ is almost strongly θ-continuous.*

Proof. Let V be regular open in Z. Then by theorem 2.2 in [2] $g^{-1}(V)$ is δ-open in Y so that $f^{-1}(g^{-1}(V))=(g\circ f)^{-1}(V)$ is θ-open in X by theorem 2.6(f). Thus, $g\circ f$ is almost strongly θ-continuous by theorem 2.6(c).

Definition 2.10. A space X is said to be *almost regular*[4] if for each $x\in X$ and each open neighborhood U of x there exists an open neighborhood V of x such that $V\subset Cl(V)\subset Int(Cl(U))$.

Theorem 2.11. *For a function $f: X\to Y$, the following statements are true:*

(a) *If X is almost regular and f is δ-continuous, then f is almost strongly θ-continuous.*

(b) *If Y is almost regular and f is continuous, then f is almost strongly θ-continuous.*

Proof. (a) Let $x\in X$ and let V be a regular open set of Y containing $f(x)$. Since f is δ-continuous, there exists an open set U_0 containing x such that $f[Int(Cl(U_0))]\subset V$. Since X is almost regular, there exists an open set U containing x such that $U\subset Cl(U)\subset Int(Cl(U_0))$. Hence $f(Cl(U))\subset f[Int(Cl(U_0))]\subset V$. This shows that f is almost strongly θ-continuous.

(b) Let $x\in X$ and let V be an open set of Y containing $f(x)$. Since Y is almost regular, there exists an open set W containing $f(x)$ such that $W\subset Cl(W)\subset Int(Cl(V))$. Since f is continuous, $x\in f^{-1}(W)\subset Cl(f^{-1}(W))\subset f^{-1}(Cl(W))\subset f^{-1}[\mathrm{Int}(Cl(V))]$.

Now let $U=f^{-1}(W)$. Then $f(Cl(U))\subset f[f^{-1}(Int(Cl(V)))]\subset Int(Cl(V))$. This shows that f is almost strongly θ-continuous.

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