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On a Generalized Volterra Equation by means of Spectral Measures

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Abstract: *In this paper we examine some properties of spectral measures and try to establish a fundamental theorem on the existence of the solution of a generalized Volterra equation in a Hilbert space as the results.*

I. Introduction

The fundamental theorem on the solution of a generalized Volterra equation (briefly, G.V.E) in a Hilbert space by means of orthoprojectors, was studied by Santis (see; (1)). In this note we are to study a fundamental theorem on the existence of the solution of a G.V.E in terms of spectral measures in a Hilbert space.

We will consider a $K\in L(H)$ (that is, algebra of bounded linear operators on H), each having the property that its eigenspaces $H_s=N_s(K)$ (that is, a eigen manifold of the operator K belonging to the eigen values) are mutually orthogonal and orthogonally reduce K. Now let the E_s be orthoprojectors on some non-zero, mutually orthogonal subspace H_s which form a complete system in H.

Let α_s be a set of spectrum. Then for a point operator K, the values of $Kx(x\in H)$ is defined by

$$Kx=\sum_s \alpha_s E_s x, \tag{1.1}$$

whenever the series (1.1) converges strongly in H. The domain Ω_k of the operator K consists of these element $x\in H$ which

$$\|Kx\|^2=\sum_s |\alpha_s|^2\|E_s x\|^2<\infty. \tag{1.2}$$

If $\{E_s\}$, $s\in J$ (J is an indexed set), is a complete system of mutually orthogonal projection operators, then for each complex function f defined on J there corresponds, by analogy to the equation (1.1), a point operator.

$$\phi(f)=\sum_s f(s)E_s=\int_{s\in J} f(s)\,dE_s \tag{1.3}$$

with respect to the finite measure

$$E(M)=\phi(\chi_M), \tag{1.4}$$

where χ_M is the characteristic function of the $M\in \boldsymbol{B}$ ($M=\bigcup_s M_s$, $M_s\in\boldsymbol{B}$). $\boldsymbol{B}$(that is, a algebra of subsets in H)

The measure $E(M)$ defined in (1.4) will be called the *measure of the spectral correspondence* ϕ or, briefly *spectral measure*. For the case of an arbitary spectral correspondence ϕ, we associate with each vector $x\in H$ a finite numerical measure

$$\mu(M;x)=(E(M)x,x)=\|E(M)x\|^2, \tag{1.5}$$

where $E(M)$ is the measure of the spectral correspondence ϕ. The measure $\mu(M;x)$, generated by the measure $E(M)$ and the vector x, will be called the spectral measure of the vector x in the correspondence ϕ. The measure $\mu(M,x)$ appears in the expressions,

$$(Kx,x)=\int_s f(s)\,d\mu(M_s:x) \tag{1.6}$$

Therefore a spectral measure in H is generally a homomorphic map of algebra $\boldsymbol{B}$ of sets into a complete algebra of proiection operators in H.

The general fundamental properties of spectral measures are omitted so that for a full discussions and proofs, we should like to refer to the reference (2) and (3). In the process of developing our discussion, we examine only some important thing necessary to developing this study further in our particular direction.

II. Some Preliminary Lemmas

A subspace $M\subset H$ orthogonally reduces $K\in L(H)$ if and only if the spectral measure $E(M)$ of the subspace M commutes with K. Indeed, let M orthogonally reduce K. Then, for any $z\in H=M\oplus M^{\perp}$, $E(M)z=x\in M$, $Kx\in M$ and $Kx^{\perp}\in M^{\perp}$. Therefore

$$KE(M)z=Kx=E(M)Kz\Rightarrow KE(M)=E(M)K \tag{2.1}$$

that is, the operator K and $E(M)$ commute. Conversely, if the condition (2.1) is satisfied, then

$$E(M)Kz=KE(M)z=Kx\Rightarrow E(M)KE(M)=KE(M) \tag{2.2}$$

Thus M is invariant under K, and whenever M is invariant, so is its orthogonal complement $M^{\perp}$, since whenever $E(M)$ commutes with K, the spectral measure $(I-E(M))$ of subspace $M^{\perp}$ also commutes with K, hence from the condition (2.2), we have

$$(I-E(M))K(I-E(M))=K(I-E(M)). \tag{2.3}$$

We sometimes call $K\in L(H)$ a ***prespectral operator*** if and only if the condition (2.1) is satified (see; (4)).

If $M_1\subset M_2$, then $E_1(M_1)E_2(M_2)=E_2(M_2)E_1(M_1)=E_1(M_1)$; it easily follows that

$$(E_2(M_2)-E_1(M_1))^2=E_2(M_2)-E_1(M_1), \tag{2.4}$$

so that $E_2(M_2)-E_1(M_1)$ is a spectral measure.

From the condition (2.2), (2.3) and (2.4), we obtain the following lemma;

Lemma 1 *If $K\in L(H)$ is a prespectral operator, then for every pair $E_1(M_1)$, $E_2(M_2)$ one has $(E_2(M_2)-E_1(M_1))K=(E_2(M_2)-(E_1(M_1))K(E_2(M_2)-E_1(M_1))$.*

Suppose that $K\in L(H))$ and $\|K\|<1$. Then $(1-K)^{-1}\in L(H)$, also

$$(1-K)^{-1}=\sum_{n=0}^{\infty}K^n \tag{2.5}$$

the convergence of the series being in the operator norm (that is in $L(H)$) and $\|(1-K)^{-1}\|\leq(1-\|K\|)^{-1}$.

Indeed, since $\|K^n\|\leqq\|K\|^n$ and $\|K\|<1$, the series in (2.5) is absolutely convergent.

And a simple calculation shows that

$$(I-K)(\sum_{n=0}^{\infty}K^n)=I=(\sum_{n=0}^{\infty}K^n)(I-K).$$

The final estimate of this is deduced from the inequalities

$$\|\sum_{n=0}^{\infty}K^n\|\leq\sum_{n=0}^{\infty}\|K^n\|\leq\sum_{n=0}^{\infty}\|K\|^n=(1-\|K\|)^{-1}.$$

We say that $K\in L(H)$ is said to satisfy a Lipschitz condition on H with Lipschitz constant l if there is a $l<\infty$ such that

$$\|Kx-Ky\|\leq l\|x-y\| \quad (\text{for } {}^{\forall}f\cdot g\in H). \tag{2.6}$$

Particularly, if K satisfies a Lipschitz condition with Lipschitz constant $l<1$, then we call $K\in L(H)$ a contraction.

Therefore, as consequences of the equations (2.5), (2.6) and (3.3), we have the following lemma;

Lemma 2 *If* $K\in L(M)$ *is contraction, then the inverse* $(I-K)^{-1}$ *exists and since it is bounded, for* $E(M)\in \boldsymbol{B}$ *one has* $E(M)(I-K)^{-1}=E(M)(I-K)^{-1}E(M)$.

III. The Main Theorems

A G.V.E is given by the form;

$$y=x-kx, \quad (\text{for } x,y\in H) \tag{3.1}$$

If the inverse bounded operator $(1-K)^{-1}$ exists, then the equation (3.1) has a unique solution $x\in H$.

Now, our starting point is to examine a propety for the resolvent $R(1;K)$ in terms of spectral measure $E(M)$. From the equation (1.3), I and K may be given by the $\phi(I)$ and $\phi(f)$, respectively. Hence, $\phi(1)-\phi(f)=\phi(1-f)$.

If a point $1\in Z$ (complex numbers) is a regular point of the operator $\phi(f)$, then

$$\inf_{(E)}|1-f(s)|>0 \tag{3.2}$$

This implies that for sufficiently small $\varepsilon>0$,

$$E(N_1^{(\varepsilon)})=0 \tag{3.3}$$

Where $N_1^{(\varepsilon)}=\{s;|1-f(s)|<\varepsilon\}$. Therefore the operator $[\phi(1-f)]^{-1}$ exists if and only if the measure $E(N_1^{(\varepsilon)})$ of the set $N_1^{(\varepsilon)}=\{s;f(s)-1=0\}$ is zero, and the resolvent of $\phi(f)$ has the integral representation,

$$R(1;K)=[\phi(1-f)]^{-1}=\int_s\frac{1}{1-f(s)}dE(M_s) \tag{3.4}$$

on its resolvent set. A point $1\in Z$ belongs to the spectrum of $\phi(f)$ if and only if,

$$\inf_{(E)}|1-f(s)|=0, \text{ w.r.t } E(M) \tag{3.5}$$

An equivalent assertion is, for all $\varepsilon>0$,

$$E(N_1^{(\varepsilon)})>0. \tag{3.6}$$

Therefore, the spectrum of the operator $\phi(f)$ coincides with the closure of set $\{f(s)\}$ with respect to $E(M)$.

In fact, if s is an eigenvalue of $\phi(f)$, then $\phi(f)-sI$ is not invertible; *i.e.* $E(N_s^{(\varepsilon)})>0$. Suppose that this condition is satisfied, and let χ_s be the characterisitic function of the set N_s. Furthermore, let x be a non-zero vector in the subspace $H(N_s)$ whose spectral measure is $E(N_s)$. Since $f(s)\chi_s(N_s)=s\chi_s=s\chi_s(N_s)$ and $\phi(s\chi_s)=sE(N_s)$,

$$\phi(f)x=\phi(f)\phi(\chi_s)x=\phi(f\chi_s)x=sE(N_s)x=sx \tag{3.7}$$

Thus, x is an eigenvector of $\phi(f)$, and

$$H(N_s)\subset H_s. \tag{3.8}$$

Conversely, if $x\in H_s$ and $x\neq 0$, then, according to the equation (1.5),

$$\int_{\beta}|s-f(s)|^2\mu(M_s : x)=\|\phi(f)x-sx\|^2=0$$

and since $|f(s)-s|>0$ on $J-N_s$, this last equation can hold only if

$$\|E(J-N_s)x\|^2=\mu(Z-N_s;x)=0$$

i.e. $E(J-N_s)x=0$. Therefore

$$x=E(N_s)x+E(J-N_s)x=E(N_s)x \tag{3.9}$$

i.e. $x\in H(N_s)$. Thus $H_s\subset H(N_s)$ and this result, when combined with the equation (3.8) shows that $H_s=H(N_s)$.

We conclude from the above results that a point $s\in Z$ bolongs to the eigenvalue spectrum of $\phi(f)$ if and only if the measure $E(N_s)$ of set $N_s=\{f(s)=s\}$ is positive. In the case, $E(N_s)$ is a spectral measure on the subspace H_s corresponding to the eigenvalue s. As consequences of the equations (1.3), (3.6) and lemma 1, we have the following theorem;

Theorem 1 *Let $K\in L(H)$ be a prespectral operator. Then necessary and sufficient conditions for K to be $K=\int_s f(s)dE(M_s)$ are that, for all $E(M)\in \boldsymbol{B}$, $E(M)K=E(M)KE(M)$ and $E(N_s^{(\varepsilon)})>0$.*

From the lemma 2, we obtain a fundamental theorem on the existence of the solution of a G.V.E as follows;

Theorem 2 *In order that exist the solution of a G.V.E, it is necessary and sufficient conditions that, for all $E(M)$, satisfy, the inverse bounded operater $(I-K)^{-1}$ exists $\Leftrightarrow$the measure of Co-spectrum, $E(N_1^{(\varepsilon)})=0$, and since $(1-K)^{-1}$ is a bounded, prespectral operator, one also has $E(M)(1-K)^{-1}=E(M)(1-K)^{-1}E(M)$.*

References

1. R.M. De santis, On a generalized Volterra equation in Hilbert space, *Proc., AMS.* **38**, pp. 563-470, 1973.
2. S.K. Berberian, *Notes on spectral theory*, D. van Nostrand. Com., Inc., 1966.
3. H.R. Dowson, *Spectral theory of linear operator*, Academic Press, 1978.
4. V. Hutson and J.S. Pym, *Applications of functional analysis and operator theory*, Academic Press, 1980.
5. Dunford, N. and Schwarz, J., *Linear operator*, Part I, II, Wiley, New York, 1963.
6. G. Bachman and L. Narici, *Functional analysis*, Academic Press, 1966.