韓國數學敎育學會誌<數學敎育>
1983. 6. Vol. XXI, No. 3

Journal of the Korea Society of Mathematical Education
Jun. 1983. Vol. XXI, No. 3

On Almost-Continuous Functions onto R_1 Spaces

By Sin Min Kang
Gyeongsang National University, Jinju, Korea

1. Introduction

R_1 spaces were introduced in 1961 by A.S. Davis [1]. In 1975, W. Dunham [3] investigated several properties of R_1 spaces and proved that R_1 spaces are weaker than Hausdorff spaces. Recently, using induced maps and natural maps, C. Dorsett [2] obtained many additional properties of R_1 spaces.

In this paper, we shall give a generalization of a result obtained by C. Dorsett [2]: *If f is a continuous open function from a space $(X,\mathcal{T})$ onto a space $(Y,\mathcal{L})$, then $(Y,\mathcal{L})$ is R_1 if and only if* $\{(x_1,x_2)\mid Cl(\{f(x_1)\})=Cl(\{f(x_2)\})\}$ *is closed in* $X\times X$.

2. Definitions and Preliminaries

Throughout this paper, spaces always mean topological spaces. Let S be a subset of a space. The closure of S and the interior of S are denoted by $\mathrm{Cl}(S)$ and $\mathrm{Int}(S)$, respectively.

Definition 2.1. A space $(X,\mathcal{T})$ is R_1 [1] iff for $x_1, x_2\in X$ such that $\mathrm{Cl}(\{x\})\neq\mathrm{Cl}(\{y\})$, there exist open sets U_1 and U_2 such that $\mathrm{Cl}(\{x_1\})\subset U_1$, $\mathrm{Cl}(\{x_2\})\subset U_2$ and $U_1\cap U_2=\phi$.

Definition 2.2. Let $(X,\mathcal{T})$ be a space and let R be the equivalence relation on X defined by x_1Rx_2 iff $\mathrm{Cl}(\{x_1\})=\mathrm{Cl}(\{x_2\})$. Then the *$T_0$-identification space* [6] of $(X,\mathcal{T})$ is $(X^*,\mathcal{T}^*)$, where X^* is the set of equivalence classes of R and $\mathcal{T}^*$ is the decomposition topology on X, which is T_0.

Definition 2.3. If f is a function from a space $(X,\mathcal{T})$ onto a space $(Y,\mathcal{L})$, then the function $f^*:(X^*,\mathcal{T}^*)\to(Y^*,\mathcal{L}^*)$ defined by $f^*(x^*)=(f(x))^*$ is the *induced map from $(X^*,\mathcal{T}^*)$ onto $(Y^*,\mathcal{L}^*)$ determined by f* [2].

In [2] and [3], C. Dorsett and W. Dunham proved the following theorems, respectively.

Theorem 2.4. *The natural map $P_X:(X,\mathcal{T})\to(X^*,\mathcal{T}^*)$ is continuous, closed, open, onto and $P_X^{-1}(P_X(U))=U$ for all $U\in\mathcal{T}$.*

Theorem 2.5. *A space $(X,\mathcal{T})$ is R_1 if and only if $(X^*,\mathcal{T}^*)$ is Hausdorff.*

3. The Main Theorems

Now, we are ready to give the main theorems.

Theorem 3.1. *If f is an almost-continuous function from a space $(X,\mathcal{T})$ onto a R_1 space $(Y,\mathcal{L})$, then $\{(x_1,x_2)\mid Cl(\{f(x_1)\})=Cl(\{f(x_2)\})\}$ is closed in $X\times X$.*

Proof. Assume that f is onto. Then f^* is onto by [2]. Let $x\in X$ and let V be an open subset of Y containing $f(x)$. Then there exists an open subset U of X containing x such that $f(U)\subset \mathrm{Int}(\mathrm{Cl}(V))$ because f is almost-continuous [5]. Put $U=P_X^{-1}(U^*)$ for any open subset U^* of X^* containing x^*. Then $f^*(U^*)=P_Y(f(P_X^{-1}(U^*)))\subset P_Y(\mathrm{Int}(\mathrm{Cl}(V)))$. By Theorem 2.4, $P_Y(\mathrm{Int}(\mathrm{Cl}(V)))\subset \mathrm{Int}(\mathrm{Cl}(P_Y(V)))$. Thus we have $f^*(U^*)\subset \mathrm{Int}(\mathrm{Cl}(P_Y(V)))$. Hence f^* is almost-continuous. Suppose that $(Y,\mathcal{L})$ is a R_1 space. Then $(Y^*,\mathcal{L}^*)$ is a Hausdorff space by Theorem 2.5. Since f^* is an almost-continuous function from $(X^*,\mathcal{T}^*)$ onto a Hausdorff space $(Y^*,\mathcal{L}^*)$, $\{(x_1^*,x_2^*) | f^*(x_1^*)=f^*(x_2^*)\}$ is closed in $X^*\times X^*$ by [4]. This shows that $\{(x_1,x_2)\,|\,\mathrm{Cl}(\{f(x_1)\})=\mathrm{Cl}(\{f(x_2)\})\}$ is closed in $X\times X$ by [2].

Theorem 3.2. *If f is an open function from a space $(X,\mathcal{T})$ onto a space $(Y,\mathcal{L})$ and if $\{(x_1,x_2)\,|\,Cl(\{f(x_1)\})=Cl(\{f(x_2)\})\}$ is closed in $X\times X$, then $(Y,\mathcal{L})$ is a R_1 space.*

Proof. Suppose that $\{(x_1,x_2)\,|\,\mathrm{Cl}(\{f(x_1)\})=\mathrm{Cl}(\{f(x_2)\})\}$ is closed in $X\times X$. Then $\{(x_1^*,x_2^*) | f^*(x_1^*)=f^*(x_2^*)\}$ is closed in $X^*\times X^*$ by [2]. Since P_Y and f are open, we have $f^*(U^*)=P_Y(f(P_X^{-1}(U^*)))\in\mathcal{L}^*$ for all $U^*\in\mathcal{T}^*$. Hence f^* is open. Moreover, since f^* is onto, $(Y^*,\mathcal{L}^*)$ is a Hausdorff space. This shows that $(Y,\mathcal{L})$ is a R_1 space.

The following Corollary 3.3 follows immediately from Theorem 3.1 and 3.2.

Corollary 3.3. ([2]) *If f is a continuous open function from a space $(X,\mathcal{T})$ onto a space $(Y,\mathcal{L})$, then $(Y,\mathcal{L})$ is R_1 if and only if $\{(x_1,x_2)\,|\,Cl(\{f(x_1)\})=Cl(\{f(x_2)\})\}$ is closed in $X\times X$.*

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