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The relationship between $\dim_A(E)$ and $\dim_B(E)$

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Introduction

Let B be a finite integral extension of a commutative ring A with identity and E a finite B-module. The purpose of this note is to study the relationship between $\dim_A(E)$ and $\dim_B(E)$ where dim means the Krull's one. The ring and dim used here will be commuative with identity and Krull's one respectively.

1. Preliminary results

Proposition 1. *Let B be integral over a ring A. If $\mathfrak{b}$ is an ideal of B and $\mathfrak{a}=A\cap\mathfrak{b}$, then $B/\mathfrak{b}$ is integral over $A/\mathfrak{a}$.*

Proof. If $x\in B$, we have, say $x^n+a_1x^{n-1}+\cdots\cdots+a_n=0$ with $a_i\in A$ if and only if $\bar{x}^n+\bar{a}_1\bar{x}^{n-1}+\cdots\cdots+\bar{a}_n=0(\text{mod } \mathfrak{b})$. Therfeore $B/\mathfrak{b}$ is integral over $A/\mathfrak{a}$.

Proposition 2. *Let A, B be as in proposition 1. Then the pair A, B satisfies incomparable and going-up.*

Proof. see ((2) p. 29).

Proposition 3. *Let the rings A, B satisfy going-up and incomparable. Then dimension of B equals the dimension of A.*

Proof. see ((2), p. 32).

Corollary. *Let A, B be as in proposition 1. Then $\dim(A/\mathfrak{a})$ equals $\dim(B/\mathfrak{b})$.*

Proof. It is clear.

Proposition 4. *Let A be a ring and E an A-module. The following results hold:*

i) $E=\sum E_i \Rightarrow supp(E)=\bigcup supp(E_i)$.

ii) *If E is finitely generated, then $supp(E)=V(annE)$.*

(and therefore a closed subset of $spec(A)$).

Proof. i) $p\in \text{supp}(E)\Rightarrow E_p=(\sum E_i)_p\neq 0$ implies that $(E_i)_p\neq 0$ for at least one i. Hence $p\in\bigcup\text{supp}(E_i)$. Reverse inclusion is obvious.

ii) Let $\{x_1, x_2, \cdots\cdots, x_n\}$ be generators of E and $E_i=Ax_i$. Then $A/\mathfrak{a}_i=E_i$ where $\mathfrak{a}_i=\text{ann}(x_i)$. Therefore $\text{supp}(E_i)=V(\mathfrak{a}_i)$. By i) $\text{supp}(E)=\bigcup_{i=1}^{n}(\text{supp}(E_i))=\bigcup_{i=1}^{n}V(\mathfrak{a}_i)=V(\bigcap\mathfrak{a}_i)=V(\text{ann}E)$.

2. Main theorem

Theorem. *Let A, B, E be as in introduction. Then $\dim_A(E)$ equals $\dim_B(E)$.*

Proof. $\dim_B(E)=\sup\{\dim B/P \mid P\in\text{spec}(B),\ E_P\neq 0\}$.

Let $n=\dim_B(E)$ and $P\in\text{spec}(B)$ be such that $E_P\neq 0$ and dim $B/P=n$. Put $p=P\cap A$, then . is an integral extension of A/p, hence $\dim B/P=\dim A/p$ by corollary.

Moreover E_P is a localization of $E_P=(A-p)^{-1}E$, therefore $E_P\neq 0$, so $\dim_A E\geqq n=\dim_B E$.

To prove the converse let $p\in\text{spec}(A)$ be such that $\dim(A/p)=\dim_A(E)$ and $E_P\neq 0$. We hav prove that there exists $P\in\text{spec}(B)$ lying over p such that $E_P\neq 0$. Replacing A, B, E by A_p, B_p, we may suppose that (A, p) is a local ring and $E\neq 0$. Then the prime ideals of B lying ov are exactly the maximal ideals of B, and since $\text{supp}_B(E)$ is a closed subset by proposition 4 t exists a maximal ideal P such that $E_P\neq 0$.

Corollary. *Let A, B, E be as in theorem, $\mathcal{C}_B$ the category of the finite B-modules and dim: $\mathcal{C}_B$ to be the Krull dimension. Then the followings are satisfied:*

i) *$\dim_A(B/\mathfrak{M})=0$ where $\mathfrak{M}$ is a maximal ideal of B.*

ii) *if $0\to E'\to E\to E''\to 0$ is an exact sequence of $\mathcal{C}_B$ then $\dim_B(E)=\max(\dim_A(E'), \dim_A(E''))$*

iii) *if $(A, \mathfrak{M})$ is a local ring and $0\to E\to E\to E/mE\to 0$ where $m\in\mathfrak{M}$ is an exact sequence of then $\dim_A(E)=1+\dim_A(E/mE)$.*

Proof. i), ii) are clear by theorem and for Proof of iii) see ((3)).

3. References

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