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A Note on the Properties of a Solvable Group

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I. Introduction

The group is the very important concept in an abstract algebra. The concept of the group is the foundation of the ring, the vector space, and the field. Therefore, I think that it is important for me to scrutinize properties of groups.

In this note, we think about that when $G_1, G_2, \cdots, G_n$, are solvable groups, the direct product of a finite number of groups is also solvable. So we will introduce some definitions and lemmata in order to prove the main theorem.

II. Definitions and Propositions

Definition 1. A group G is said to be *abelian* if its binary operation is commutative.
$ab=ba$ for all $a, b \in G$.

Proposition 2. *If N is a subgroup of G, then $NN=N$.*

Proof $x \in N \Rightarrow xx \in N$. But $xx \in NN$. Therefore, $N \subset NN$. (1)
$y \in NN \Rightarrow y=xx$ for all $x \in N$. But $xx \in N$ for all $x \in N$, $y \in N$. Therefore $NN \subset N$. (2)
By (1) and (2), $NN=N$.

Definition 3. A subgroup H of a group G is *normal* if it is the kernel of some homomorphism of G into some group.

Proposition 4. *A subgroup H of a group G is normal if and only if $gHg^{-1}=H$ for all $g \in G$.*

Proof Since H is the kernel of some homomorphism f, $f(x)=e'$ for all $x \in H$ where $f: G \to G'$. e'; a unit element in G'. Therefore, $f(gxg^{-1})=f(g)f(x)f(g)^{-1}=f(g)e'f(g)^{-1}=e'$.
$\therefore gxg^{-1} \in H$: *i.e.* $gHg^{-1}=H$.

Conversely, for all $g \in G$, $x \in H$, $gHg^{-1}=H$ if and only if $f(x)=e'$. Therefore, if $gHg^{-1}=H$ for all $g \in G$, then H is normal.

Definition 5. If N is a normal subgroup of G, then $G/N=\{Na \mid a \in G\}$ is a *factor group* or a *quotient group*.

Definition 6. Let H be a subgroup of G. $aH=\{ax \mid x \in H\}$ is a *left coset* of H in G.

Definition 7. A group G is *solvable* if there exists a sequence of subgroups $G=H_0 \supset H_1 \supset H_2 \cdots \supset H_m=\{e\}$ such that H_i is normal in H_{i-1} and such that the factor group H_{i-1}/Hi is abelian, for $i=1, 2, \cdots, m$.

Definition 8. Let A and B be groups. $G=A \times B=\{(a, b) \mid a \in A, b \in B\}$ is said to the *direct product* of A and B. For (a_1, b_1), (a_2, b_2) in G, their product $(a_1, b_1)(a_2, b_2)=(a_1a_2, b_1b_2)$.

III. Main Theorem

In order to prove the main theorem, some lemmata are needed.

Lemma 1. *Let G_1 and G_2 be groups and H_1 and H_2 subgroups of G_1, G_2 respectively. Then $H_1 \times H_2$ is also a subgroup of $G_1 \times G_2$.*

Proof (a_1, b_1), (a_2, b_2), $(a_3, b_3) \in H_1 \times H_2$ where $a_1, a_2, a_3 \in H_1$ and $b_1, b_2, b_3 \in H_2$.

1. $(a_1, b_1)\{(a_2, b_2)(a_3, b_3)\} = (a_1, b_1)(a_2a_3,\ b_2b_3) = (a_1(a_2a_3),\ b_1(b_2b_3)) = ((a_1, a_2)a_3,\ (b_1, b_2)b_3) = \{(a_1, b_1)(a_2, b_2)\}(a_3b_3)$.
2. $(a_1, b_1)(a_2, b_2) = (a_1a_2,\ b_1b_2) \in H_1 \times H_2$.
3. $(a_1, b_1)(e_1, e_2) = (a_1e_1,\ b_1e_2) = (a_1, b_1)$ where e_1, e_2 are unit elements in H_1 and H_2 respectively.
4. $(a_1, b_1)(a_1^{-1}, b_1^{-1}) = (a_1a_1^{-1}, b_1b_1^{-1}) = (e_1, e_2)$

Lemma 2. *Let G_1 and G_2 be a group and let N_1 and N_2 be normal subgroups of G_1 and G_2 respectively. Then $N_1 \times N_2$ is a normal subgroups of $G_1 \times G_2$.*

Proof Let $x_1 \in N_1$, $y_1 \in G_1$ and $x_2 \in N_2$, $y_2 \in G_2$. Then $y_1x_1y_1^{-1} \in N_1$, and $y_2x_2y_2^{-1} \in N_2$

$$(y_1, y_2)(x_1, x_2)(y_1, y_2)^{-1} = (y_1x_1, y_2x_2)(y_1^{-1}, y_2^{-1}) = (y_1x_1y_1^{-1}, y_2x_2y_2^{-1}) \in N_1 \times N_2.$$

For $(y_1, y_2) \in G_1 \times G_2$, $(y_1, y_2)N_1 \times N_2\ (y_1, y_2)^{-1} = N_1 \times N_2$.

Therefore, $N_1 \times N_2$ is normal in $G_1 \times G_2$.

This result can be extended to the direct product of a finite number of groups.

The main theorem: *If $G_1, G_1, \cdots, G_n$ are solvable groups, then the direct product $G_1 \times G_2 \times \cdots \times G_n$ is solvable.*

Proof Let G_1 and G_2 be solvable groups.

There exist two sequences $G_1 = H_{11} \supset H_{12} \supset \cdots \supset H_{1m} = \{e_1\}$ and $G_2 = H_{21} \supset H_{22} \supset \cdots \supset H_{2m} = \{e_2\}$ such that H_{1i} is normal in H_{1i-1} and the factor group H_{1i-1}/H_{1i} is abelian and H_{2i} is normal in H_{2i-1} and H_{2i-1}/H_{2i} is abelian.

Therefore, there exists a sequence $G_1 \times G_2 = H_{11} \times H_{21} \supset H_{12} \times H_{22} \supset \cdots \supset H_{1m} \times H_{2m} = \{(e_1, e_2)\}$, where $H_{1i} \times H_{2i}$ is normal in $H_{1i-1} \times H_{2i-1}$.

And we must show that the factor group $H_{1i-1} \times H_{2i-1}/H_{1i} \times H_{2i}$ is abelian. Since H_{1i-1}/H_{1i} is abelian, $xyH_{1i} = yxH_{1i}$ for $x, y \in H_{1i-1}$.

Let $(x, z)(y, w) \in H_{1i-1} \times H_{2i-1}$

$(x, z)(H_{1i} \times H_{2i})(y, w)(H_{1i} \times H_{2i})$

$= (x, z)(y, w)(H_{1i} \times H_{2i})$ (by proposition 2)

$= (xy, zw)(H_{1i} \times H_{2i})$ (by proposition 8)

$= xyH_{1i} \times zwH_{2i}$

$= yxH_{1i} \times wzH_{2i}$ (H_{1i} and H_{2i} are normal)

$= (yx, wz)(H_{1i} \times H_{2i}) = (y, w)(x, z)(H_{1i} \times H_{2i})(H_{1i} \times H_{2i})$

$= (y, w)(H_{1i} \times H_{2i})\ (x, z)(H_{1i} \times H_{2i})$

$(H_{1i-1} \times H_{2i-1})/H_{1i} \times H_{2i}$ is abelian.

Therefore $G_1 \times G_2$ is solvable. The proof that $G_1 \times G_2 \times \cdots \times G_n$ is solvable is trivial by induction.

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