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Nonhomogeneous Equations without Nontrivial C^1 Solutions

By Yong Jing Suk

Kang Won National University, Chuncheon, Korea

In this paper, we give the two examples with which the Mizohata operator has no C^1 solution in any neighborhood of the origin and characterize the C^1 solutions of the perturbed Mizohata operators.

1. Preliminary

Theorem 1.1. *Let W be a connected open subset of R^2 invariant under the symmetry $(x,t)\mapsto(x,-t)$. Of u is a C^1 solution satisfies $Mu=0$ in W, we have $u(x,t)=u(x,-t)$ in W.*

Proof. See F. Treves [1].

2. Main Theorems

Let D_n $(n=1,2,\dots)$ be an arbitrary sequence of closed nonoverlapping discs in the right half of the (x,t) plane, $t>0$, with centres $(0,t_n)$, $t_n>0$ and $t_n\to 0$.

Let $f(x,t)$ be an arbitrary chosen C^∞ function with compact support which is such that

$$\iint_{D_n} f\,dx\,dt\neq 0 \text{ for } n=1,2,\dots$$

Theorem 2.1. *If f satisfies the conditions above, there is no C^1 solution of $Mu=f$ in any neighborhood of the origin. Here $M=\frac{\partial}{\partial t}+it\frac{\partial}{\partial x}$.*

Proof. See L. Nirenberg [4].

Corollary 2.2. *Let f be a function in Theorem 2.1. If u is a C^1 solution satisfying $Mu=fu$ in an arbitrary neighborhood of the origin, then $u(0,0)=0$.*

Proof. Suppose $u(0,0)\neq 0$. $u\neq 0$ in some open subset D of the origin. Let $w=\log u$, $Mw=\frac{1}{u}Mu=\frac{1}{u}fu=f$ in D. W is a C^1 solution satisfying $Mu=f$. It contradicts to the choice of f in D Theorem 2.1. (Q.E.D)

Let $K_{m,n,p}(m,n,p$ are positive numbes) be a triple sequence of compact sets in the upper half plane $t>0$, such that the following is true;

(2.1) the projections on the x-axis $t=0$ of the $K_{m,n,p}$ are pairwise disjoint;

(2.2) for m,n fixed, $\lim_{p\to\infty} K_{m,n,p}=(x_{m,n},\ t_{m,n})$ with $t_{m,n}>0$;

(2.3) for m fixed, $\lim_{n\to\infty}(x_{m,n},\ t_{m,n})$ with $t_m>0$;

(2.4) $(x_m,t_m)\to(0,0)$ as $m\to\infty$

We take then $g\in C^\infty(R^2)$ with the following properties;

(1) $g \geq 0$ everywhere

(2) $g \equiv 0$ outside $\bigcup_{m,n,p} K_{m,n,p}$

(3) $g > 0$ at some point of each $K_{m,n,p}$.

Theorem 2.3. *If g satisfies the conditions above, there is no C^1 solution satisfying $Mu = g$ in a*
neighborhood of the origin.

If u is a C^1 solution satisfying $Mu = gu$ in any neighborhood W of the origin, $u \equiv 0$ in W.

Proof. There are numbers m_0, n_0, p_0 large enough such that, for $m > m_0$, $n > n_0$, $p > p_0$, the foll
wing is true; Let $R_{m,n,p}$ be the rectangle $a_{m,n,p} < x < b_{m,n,p}$, $|t| < T$. The numbers $a_{m,n,p}$, $b_{m,n,p}$ a
$T > 0$ can be chosen so that $K_{m,n,p} \subset R_{m,n,p} \subset W$, $R_{m,n,p} \cap K_{m',n',p'} = \emptyset$ if $(m, n, p) \neq (m', n', p')$. N
that $K_{m,n,p} \cap K_{m',n',p'} = \emptyset$ if $(m, n, p) \neq (m', n', p')$.

Let then $K_{m,n,p}$ denote the image of $K_{m,n,p}$ under the map $(x, t) \mapsto (x, -t)$. Then $W_{m,n,p} = R_{m,}$
$-(K_{m,n,p} \cup K_{m,n,p})$ and let $W^0_{m,n,p}$ be the outer connected component of $W_{m,n,p}$. Note that $W^0_{m,}$
is open, conneted and symmetric in t.

Suppose then that $Mu = g$ in W. Since $g \equiv 0$ in $R_{m,n,p} - K_{m,n,p}$, we have $Mu = 0$ in $W^0_{m,n,p}$.

Let r be a smooth closed curve in $W^0_{m,n,p} \cap \{(x, t) \mid t > 0\}$ whose interior contains $K_{m,n,p}$.

We claim that

(2.5) $\int_r u(x, t)(dx - it\ dt) = 0$. Since $u(x, t) = u(x, -t)$ in $W^0_{m,n,p}$ by Theorem 1.1.

$\int_r u(x, t)(dx - it\ dt) = \int_{\check{r}} u(x, t)(dx - it\ dt) = \int_{\tilde{r}} \tilde{u}(z) dz$, where $\check{r}$ is the image of r under the m
$(x, t) \mapsto (x, -t)$, $\tilde{r} = z(r) = z(\check{r})$ and $\tilde{u}(z) = u$ (Re z, $\sqrt{-2}$ Im z) via the map $(x, t) \mapsto z = x - i\frac{t^2}{2}$.

$Mu = Mz \frac{\partial \tilde{u}}{\partial z} + m\bar{z} \frac{\partial \tilde{u}}{\partial \bar{z}} = 2it \frac{\partial \tilde{u}}{\partial \bar{z}}$. Since $Mz = 0$ and $Mu = 0$ in $W_{m,n,p} - K_{m,n,p}$, $\frac{\partial \tilde{u}}{\partial \bar{z}} = 0$ in a simp
connected domain, the image W' of $W_{m,n,p} - K_{m,n,p}$ via the map z and r is a smooth closed cur
in W'. By Cauchy theorem, $\int_{\tilde{r}} \tilde{u}(z) dz = 0$. Thus our claim is proved.

By Stoke's Theorem,

$$\int_r u(x, t)(dx - itdt) = \iint_{\text{Int } r} d[u(dx - itdt)] = -\iint_{\text{Int } r} \left(\frac{\partial u}{\partial t} + it \frac{\partial u}{\partial x} \right) dxdt = -\iint_{\text{Int } r} Mu\ dx\ dt = -\iint_{\text{Int } r} g\ dx$$

$$= -\iint_{K_{m,n,p}} g\ dx\ dt \neq 0.$$

Since $g \equiv 0$ outside $K_{m,n,p}$ and $g > 0$ at some point of each $K_{m,n,p}$. It is a contradiction to (2.5
Thus we get the first part of the Theorem.

Now we shall give the second part of the Theorem.

$$\iint_{\text{Int } r} Mu\ dx\ dt = \iint_{\text{Int } r} g\ dx\ dt = \iint_{K_{m,n,p}} gu\ dx\ dt.$$

In conclusion,

$$\iint_{K_{m,n,p}} gu\ dx\ dt = 0. \tag{2.6}$$

Now suppose we had

$$u(x_{m,n},\ t_{m,n}) \neq 0. \tag{2.7}$$

Then, for p large enough, in $K_{m,n,p}$ Arg u would be very colse to Arg u $(x_{m,n}, t_{m,n})$ by the continuity of u. Since $g>0$ in some subset of $K_{m,n,p}$, (2.6) would be impossible. Therefore (2.7) is not true.

Therefore u has a infinite sequence of zeros, converging to (x_m, t_m). But in the neighborhood of (x_m, t_m) (Since $t_m>0$), $L=\frac{\partial}{\partial t}+it\frac{\partial}{\partial x}-g(x,t)$ is elliptic. We can choose a suitable local coordinates ξ, η such that $L=w(\xi,\eta)\left(\frac{\partial}{\partial \xi}+i\frac{\partial}{\partial \eta}\right)$ with $w\neq 0$, according to the Newlander-Nirenberg Theorem. Therefore u is a holomorphic function of $\zeta=\xi+i\eta$, $u\equiv 0$ in a full neighborhood of (x_m, t_m), and therefore in the entire set $W^+=\{(x,t)\in W \mid t>0\}$.

But then $u\equiv 0$ in the open set $W^0_{m,n,p}\cap\{(x,t)\mid t<0\}$, hence in W^- by Theorem 1.1, hence in W by the continuity of u. (Q.E.D)

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