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On Linear Functionals on Saks Spaces

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The purpose of this note is to show some basic properties of γ-linear functionals on Saks spaces. We begin with some definitions given in [2], [3], and [4].

A Frecht norm $|\ |$ on a linear set X is a real-valued non-negative function with the following properties:

(1) $|x|=0$ if and only if $x=0$,

(2) $|x+y|\leq|x|+|y|$ for all x, y in X,

(3) if $\{a_n\}$ is a sequence of real numbers converging to a real number a and $\{x_n\}$ is a sequence of points of X with $|x_n-x|\to 0$, then $|a_nx_n-ax|\to 0$.

It is called a B-norm if the condition (3) is replaced by

(4) $|ax|=|a||x|$ where a is any real number and x is any element of X.

Let $|\ |_1$ and $|\ |_2$ be two norms (B or F) defined on X. We define $|\ |_1\geqq|\ |_2$ if $|x_n|_1\to 0$ implies $|x_n|_2\to 0$. When $|\ |_2\geqq|\ |_1$ and $|\ |_1\geqq|\ |_2$, we say that $|\ |_1$ is equivalent to $|\ |_2$ and write $|\ |_1\sim|\ |_2$.

A two norm space is a linear set X with two norms, a B-norm $|\ |_1$ and an F-norm $|\ |_2$. A sequence $\{x_n\}$ of points in a two norm space $(X, |\ |_1, |\ |_2)$ is said to be *γ-convergent* to x in X, written $x_n\xrightarrow{\gamma}x$, if lim sup $x_n<\infty$ and $\lim_n|x_n-x|_2=0$.

A sequence $\{x_n\}$ in a two norm space is said to be *γ-Cauchy* if $(x_{p_n}-x_{q_n})\to 0$ as $p_n, q_n\to\infty$. A two norm space $X_s=(X, |\ |_1, |\ |_2)$ is called *γ-complete* if for every *γ-Cauchy* sequence $\{x_n\}$ in X_s there exists an x in X_s such that $x_n\xrightarrow{\gamma}x$.

A γ-linear functional f on two norm space is a real-valued function on X_s such that

(1) $f(ax+by)=af(x)+bf(y)$, for every real numbers a, b and any x, y in X_s,

(2) if $x_n\xrightarrow{\gamma}x$, then $f(x_n)\to f(x)$.

The set of all γ-linear functionals on X_s will be denoted by X_s^*. It is easy to see that X_s^* is a linear set.

Let X be a linear set and suppose that $|\ |_1$ is a B-norm, and $|\ |^*$ is an F-norm on X. Let $X_s=\{x\in X: |x|_1<1\}$ and define $d(x,y)=|x-y|^*$ in X_s. Then d is a metric on X_s and the metric space (X_s, d) will be called a ***Saks set***. If (X_s, d) is complete, it will be called a ***Saks space***. We will denote (X_s, d) by $(X, |\ |_1, |\ |^*)$.

Banach ([1], p. 243) has defined the following

Definition 1. Let $\{X_k\}$ be a class of Banach spaces. Define

$$l(X_1, X_2, \dots)=\{\{x_k\} : x_k \in X_k \text{ for each } k \text{ and } \sum_{k=1}^{\infty}|x_k|_{X_k}<\infty\}.$$

If we define vector addition in the usual way the above set is linear. We often denote $l(X_1, X_2, \dots)$ by $l(\{X_k\})$. Define $|x|_{l(\{X_k\})}=\sum_{i=1}^{\infty}|x_i|_{X_i}$ for $x=\{x_i\}$. Then it is easy to see that the space $(l\{X_k\}, \; | \; |_{l(\{X_k\})})$ is a Banach space. We prove the following

Theorem 1. *The dual of the space* $(l(\{X_i\}), \; | \; |_{(\{X_i\})})$ *is the space* $(m(\{X_i^*\}), \; | \; |_{m(\{X_i^*\})})$ *where* $m(\{X_i^*\})=\{\{x_k\} : x_k \in X_k^*$ *for each* k *and* $sup_k|x_k|_{X_k^*}<\infty\}$, *and* $|x|_{m(\{X_k^*\})}=sup_k|x_k|_{X_k^*}$, $x=\{x_k\}$.

Proof. We can show that every linear functional on $l(\{X_i\})$ is associated with a unique sequence $g=\{g_i\}$, $g_i \in X_i^*$ for each i. For $x=\{x_i\} \in l(\{X_i\})$, we have $\lim_{n\to\infty}|x-\sum_{i=1}^{n}x_i|_{l(\{X_i\})}=0$ where x_i represents the sequence $(0, 0, \dots, x_i, 0, \dots)$ and the space X_i is identified with the space $(0, 0, \dots, X_i, 0, \dots)$. If $f \in [l(\{X_i\})]^*$, then $f(x)=\sum_{i=1}^{\infty} f(x_i)=\sum_{i=1}^{\infty} f|_{X_i}(x_i)=\sum_{i=1}^{\infty} g_i(x_i)$, where $g_i \in X_i^*$ and $g_i=f|_{X_i}$. It is easy to see that $[U\{X\}_i)]^*$ is isometrically isomorphic to $(m(\{X_i\}), \; | \; |_{m(\{X_i^*\})})$.

We define $\| \{x_k\} \|_s=\sum_{i=1}^{\infty}\frac{1}{2^i}|x_i|_{X_i^*}/(1+|x_i|_{X_i^*})$ and denote the Saks set $(m(\{X_k^*\}), \; | \; |_{m(\{X_k^*\})}, \; \| \; \|_s)$ by $(m_s(\{X_k^*\}), \; d)$.

Theorem 2. *The Saks set* $(m_s(\{X_k^*\}), d)$ *is a complete metric space, that is,* $(m_s(\{X_k^*\}), \; d)$ *is a Saks space.*

Proof. Let $\{x_n\}$, $x_n=\{x_{n,k}\}$ be a Cauchy sequence of points from $m_s(\{X_k^*\})$. Then

$$|x_{n,i}-x_{m,i}|_{X_i^*}\to 0 \text{ as } n, m\to\infty, \text{ for each } i.$$

Since each X_i is complete, there is $z_i \in X_i^*$ with $|z_i|_{X_i^*}\leqq 1$ such that

$$|x_{n,i}-z_i|_{X_i^*}\to 0 \text{ as } n\to\infty.$$

Hence $z=\{z_i\} \in m_s(\{X_k^*\})$. Since coordinatewise convergence is equivalent to $\| \; \|_s$ convergence, we have $\|x_n-z\|_s\to 0$ as $n\to\infty$. Hence $(m_s(\{X_k^*\}), \; d)$ is complete.

Theorem 3. *The space* $l(\{X_k^{**}\})$ *can be identified with a subset of* $[m_s(\{X_k^*\})]^*$, *the space of* γ*-linear functionals on* $m_s(\{X_k^*\})$.

Proof. We show that if $f \in [m_s(\{X_k^*\})]^*$ then f can be uniquely associated with a sequence $\{f_i\}$, $f_i \in X_i^{**}$. If x_i represents the sequence $(0, 0, \dots, x_i, 0, \dots)$ and the space X_i^* is identified with the space $(0, 0, \dots, X_i^*, 0, \dots)$, we have

$$\lim_{n\to\infty}\|x-\sum_{i=1}^{n}x_i\|_s=0 \text{ for } x=\{x_i\} \in m_s(\{X_k^*\}).$$

Hence, for $f \in [m_s(\{X_k^*\})]^*$, we have

$$f(x)=\sum_{i=1}^{\infty}f(x_i)=\sum_{i=1}^{\infty} f|_{X_i^*}(x_i)=\sum_{i=1}^{\infty}f_i(x_i),$$

where $f_i=f|_{X_i^*} \in X_i^{**}$ and f_i represents the sequence $(0, 0, \dots, f_i, 0, \dots)$.

Now we show that $f \in [m_s(\{X_k^*\})]^*$ for $f=\{f_i\} \in (\{X_k^{**}\})$. Let $\{x_n=\{x_{n,k}\}\}$ be a sequence of points from $m_s(\{X_k^*\})$ and $\|x_n\|_s\to 0$.

Since $f \in l(\{X_k^{**}\})$, there is N_1 such that $\sum_{k=n}^{\infty} |f_k|_{X_k^{**}}<\epsilon/2$ for $n\geqq N_1$, and there is $\delta=\delta(\epsilon)$ such that for all n with $\sup_{1\leq i\leq N_1} |x_{n,i}|_{X_i}^* < \delta$, we have $|\sum_{i=1}^{N_1} f_i(x_{n,i})| < \epsilon/2$.

Since $\|x_n\|_s \to 0$, there exists N_2 such that

$$\sup_{1\leq i\leq N_i} |x_{n,i}|_{X_i}^* < \delta \text{ for } n > N_2.$$

Hence,

$$\left|\sum_{i=1}^{N_1} f_i(x_{n,i})\right| < \varepsilon/2 \text{ for } n > N_2.$$

Thus we have, for $n > N_2$,

$$\left| f(x_n) \right| = \left| \sum_{i=1}^{\infty} f_i(x_{n,i}) \right| \leq \left| \sum_{i=1}^{N_1} f_i(x_{n,i}) \right| + \left| \sum_{i=N_1+1}^{\infty} f_i(x_{n,i}) \right|$$

$$< \varepsilon/2 + \sum_{i=N_1+1}^{\infty} |f_i|_{X_i}^{**}, \quad |x_{n,i}|_{X_i}^*$$

$$< \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

To obtain the last inequality we have used the fact that if $x_n \in [m_s(\{X_k^*\})]^*$ then $|x_{n,i}|_{X_i}^* \leqq 1$ for each i. Thus $f \in [m_s(\{X_k^*\})]^*$.

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