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On Weak Compactness in Spaces of Continuous Functions

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Weakly compact subsets (compact with respect to the weak topology of a Banach space or of a locally convex space) play an important role in many questions of analysis. Especially the various questions of weak compactness in spaces of continuous functions are useful and interesting facts in functional analysis. This paper will be concerned with such results and show that whether some of them are hold or not by exchanging space or conditions. In this paper, the space X means the compact Hausdorff space with the conjugates X^* and X^{**}, $C(X)$ the space of continuous functions on X, $C^b(X)$ the space of bounded continuous functions on X, $C(X,Z)$ the space of continuous functions from X to Z where X, Z are Hausdorff topological spaces and W_x the (Hausdorff) topology of pointwise convergence in X on $C(X,Z)$.

If X is countably compact and $E\subset C(X)$ is W_x-relatively countably compact, then the sequences $\langle x_n\rangle \in X$ and $\langle f_m\rangle \in E$ have cluster points $x\in X$ and $f\in C(X,Z)$ respectively. Therefore $f(x_n)$ is a cluster point of $(f_m(x_n))_m$, but if $\lim_m f_m(x_n)$ exists, the only cluster point of this sequence is the limit: $\lim_m f_m(x_n)=f(x_n)$. This argument repeated implies that $\lim_n \lim_m f_m(x_n)=f(x)=\lim_m \lim_n f_m(x_n)$ (the interchangable double-limit property, denote by $E\sim X$ (in Z)) if all these limit exist. Therefore, with the Šmulian theorem, the following property holds.

Proposition 1. *Let D be a dense subset of a countably compact space, Z a compact metric space and $E\subset C(X,Z)$. The following are equivalent.*

(1) *E is W_x-relatively countably compact in $C(X,Z)$.*

(2) *E and D have the interchangable double-limit property in Z.*

(3) *E is W_x-relatively compact in $C(X,Z)$.*

The implication (2)⇒(3) does not hold provided X is compact, Z metric and only locally compact since $\mathbf{R}\subset C([0,1],\mathbf{R})$. By specializing this proposition to real-valued functions $C(X,\mathbf{R})$ we obtain the following.

Corollary 1.1. *Let X be a topological space, $D\subset X$ dense*

(1) *If $E\subset C^b(X)$ is uniformly bounded and $E\sim D$ (in $\mathbf{R}$), then E is W_x-relatively compact in $C^b(X)$.*

(2) *If $E\subset C(X)$ is pointwise bounded and $E\sim D$ (in $\mathbf{R}$), then E is W_x-relatively compact in $C(X)$.*

Proof: (1) If $|f(X)|\leqslant a$ for all $x\in X$ and $f\in E$ then, by (Proposition 1), E is W_x-relatively compact in $C(X,[-a,a])$. (2) Consider $C(X)$ as a subset (in its topology) of $C(X,\mathbf{R})$ and notice that pointwise limits of functions in E are everywhere finite: This means that the W_x-closure of E in $C(X, \boldsymbol{R})$, which is compact by (Proposition 1), is already in $C(X)$. ///

Corollary 1.2. *If X is countably compact, Z metric, then a subset $A \subset C(X, Z)$ is W_x-relatively countably compact iff it is W_x-relatively compact.*

Proposition 2. *A sequence $f_n \in C(X)$ converges weakly to $f_0 \in C(X)$ iff the x_n are uniformly bounded and converge pointwise to x_0.*

Proof: Let $\delta_{x_0}(f) = f(x_0)$ be a continuous linear functional on $C(X)$. Then the necessity follows directly from $\delta_{x_0}(f_n) \to \delta_{x_0}(f_0)$ that f_n converges pointwise to f_0. Since the sequence f_n is strongly bounded subset of $C(X)$, it must also be uniformly bounded on X, so that the f_n are uniformly bounded. To prove the sufficiency we use the following theorem of Lebesgue as lemma: Let μ be a positive measure on X and $L^1{}_\mu$ the space of absolutely integrable functions with norm $\|f\| = \int |f| d\mu$. If the sequence $h_n \in L^1{}_\mu$ converges μ-almost everywhere to h_0 and if $|h_n| \leqslant g$ μ-almost everywhere for some g in $L_\mu{}^1$, then $h_0 \in L^1{}_\mu$ and $\int |h_n - h_0| d\mu \to 0$. Since $C(X)$ is a space of every $L^1{}_\mu$, the weak convergence of f_n follows for every positive μ, and so for every measure on X. ///

Corollary 2.1. *If a sequence $\langle f_n \rangle$ in $C[a, b]$ converges weakly to $\langle f \rangle$ in $C[a, b]$, then the sequence is uniformly bounded and for $a \leqslant x \leqslant b$, $\lim_{n \to \infty} f_n(X) = f(x)$.*

Proposition 3. *A subset $E \subset C[a, b]$ is weakly compact iff it is uniformly bounded and $W_{[a,b]}$-compact.*

Proof: The weaker topology is finer than $W_{[a,b]}$ and, by Mackey's theorem on conjugate pairs, every weakly bounded set is bounded in $C[a, b]$, that is uniformly bounded. On the other hand, by the fact that both topologies are angelic (a topological Hausdorff space X is called angelic if for every relatively compact set $E \subset X$, E is relatively compact and there is a sequence in E which converges to x for each x in $\bar{E}$, it is enough to show that every uniformly bounded pointwise convergent sequence is weakly convergent. But observing that every φ in coniugate $C^*[a, b]$ of $C[a, b]$ is represented by a measure this is true by Lebesgue's dominated convergence theorem. ///

Corollary 3.1. *$E \subset C^b[a, b]$ is weakly relatively compact iff it is uniformly bounded on $[a, b]$ and has the interchangable double-limit with $[a, b]$.*

Proof: Since $[a, b]$ is compact and completely regular, by (Proposition 1) and (Corollary 1.1) it is obvious. ///

Corollary 3.2. *If $E \subset C(X)$ is convex and W_x-relatively countably compact, then E is uniformly bounded on all bounding subsets $Y \subset X$.*

Proof: The set $B = \{f \in C(X) : \sup_{x \in Y} |f(x)| \leqslant 1\}$ is W_x-closed, absolutely convex and absorbing, therefore an W_x-barrel. So that $E \subset \lambda B$ for some $\lambda > 0$. Hence E is uniformly bounded on Y. ///

Corollary 3.3. *A subset $E \subset C(X)$ is weakly relatively compact iff it is bounded and W_x-relatively compact.*

Proof: The condition is necessary, for a relatively weakly compact set is also relatively compact under the coarser topology W_x. On the other hand suppose that E is bounded and relatively W_x-

compact. In order to show that E is relatively weakly compact it is sufficient, by Eberlein's theorem, to show that every sequence f_n in E contains a weakly convergent subsequence. Since every relatively countably W_x-compact subset of $C(X)$ is W_x-relatively sequentially compact f_n has a W_x-convergent subsequence, and this converges weakly by Proposition 2. ///

Obviously, by previous properties of weak compactness in $C(X)$, an application of the Eberlein-Šmulian theorem characterizes weakly compact subset of $C[0, 1]$ as following.

Corollary. *A subset $E \subset C[0, 1]$ is weakly compact iff it is weakly closed, norm bounded and every sequence $\langle f_n \rangle$ in E has a subsequence $\langle f_{n_k} \rangle$ such that $\lim_{k\to\infty} f_{n_k}(x) = f(x)$ for some $f \in E$ and all $x \in [0, 1]$.*

Remark 1. Since a compact Hausdorff space X need not be sequentially compact the following example shows that the Eberlein-Šmulian theorem is false for weak* compact subsets. If $P : X \to C^*(X)$ be defined by $P(x)(f) = f(x)$ for such X, then P is a homeomorphism from X onto $P(X) \subset C^*(X)$ with the weak* topology.

Remark 2. Generally, a reflexive Banach space X is weakly sequentially complete. But $C[0, 1]$ is not weakly sequentially complete. For if we let $x_n(t) = (1-t)^n$, then $x_n(t)$ cannot converge pointwise to a continuous function and hence by Corollary 2.1 cannot converge weakly. But $C^*[0, 1]$ is weakly sequentially complete (see [7], p. 120). Thus we can see that $C[0, 1]$ is not reflexive.

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