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A Note on the Spectrum of any Self-adjoint Extension

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Abstract. In this note we consider properties for the discreteness of the spectrum of second-order differential operators. If we give some conditions, then the spectrum of any self-adjoint extension of A_0, $A_0u=a[u]$, $D(A_0)=C_0^\infty(0.1)$ is discrete.

1. Introduction

The spectrum of an operator A is discrete if and only if $\sigma_e(A)$ is empty. Let $p(x)$ and $q(x)$ be real-valued functions defined on the interval $w=[x_1, x_2]$, with length $|w|$ where $p(x)\geqslant 0$ and $p^{-1}(x)$ and $q(x)$ are integrable over w. Let $\mu w=\frac{1}{|w|}\int_w q(x)\,dx$ denote the mean value of $q(x)$ on w, and let either $q(x)\geqslant 0$ or $q(x)\leqslant 0$ on w. Then

$$\int_w (p(x)|u'|^2+q(x)|u|^2)dx\geqslant\mu_w(1+|w|\mu_w\int_w p^{-1}(x)dx)^{-1}\|u\|^2_w$$

for every (complex valued) function $u(x)\in C^1[x_1, x_2]$.
We assume that the $a_k(x)$ satisfy the following conditions

i) $a_k(x)\in W_2^k(0, X)$, $0\leqslant K\leqslant n$, for all $X>0$.

ii) $a_n(x)>0$, $0\leqslant x\leqslant\infty$.

The first condition means that $a_k(x)$ belongs to the Sovolev-space $W_2^k(0, X)$ for all $x, 0<x<X$. We consider the spectrum of differential operators.

2. Auxiliary results

Now, we list some lemmas necessary to developing our discussion further in our particular direction.

Lemma 1. *If $a_k(x)\geqslant 0$ on the interval $w=[x_1, x_2]$ for all K, then the inequalities*

$$\sum_{k=r}^{s}\int_w a_k(x)|u^{(k)}|^2dx\geq\left(w^{2(s-r)}\rho_{w,s}{}^{-1}+\sum_{k=r}^{s-1}|w|^{2(k-r)}\mu_{w,k}\right)^{-1}\|u^{(r)}\|^2w,\ 0\leqslant r<s\leqslant n,$$

hold for all $u(x)\in C^n[x_1, x_2]$, where

$$\mu_{w,k}=\frac{1}{|w|}\int_w a_k(x)dx \text{ and } \rho_{w,k}=\left(\frac{1}{|w|}\int_w\frac{dx}{a_k(x)}\right)^{-1},\ 0\leqslant k\leqslant n.$$

Next we consider necessary and sufficient condition for the discreteness of the spectrum of second-order differential operators where the behaviour of the highest is not restricted by a power x^α.

Lemma 2. *Let the coefficients of the Sturm-Liouville differential expression*

$$a[\cdot]=-\frac{d}{dx}p(x)\frac{d}{dx}+q(x),\ 0\leq x<\infty$$

satisfy the following conditions.

i) $p(x)\in W_2^1(0,X)$ *and* $q(x)\in L_2(0,X)$ *for all* $X>0$.

ii) $p(x)>0,\ 0\leqslant x<\infty$, *and* $\lim\limits_{x\to\infty} \inf q(x)>-\infty$.

iii) There are positive constants h_0, c_0, *and* C_0 *such that*

$$C_0 p(x_1)\leqslant p(x_2)\leqslant C_0 p(x_1) \ \ for\ 0\leqslant x_1\leqslant x_2\leqslant x_1+h_0\ p^{1/2}(x_1).$$

Then the spectrum of any self-adjoint exclusion of A_0,

$$A_0 u=a[u],\ \ D(A_0)=C_0^{\infty}(0,\infty),$$

is discrete if and onlf if

$$\lim_{x\to\infty}\frac{1}{p^{1/2}(x)}\int_x^{x+hp^{1/2}(x)} q(t)dt=\infty \ \ for\ each\ fixed\ 0<h<1.$$

Lemma 3. *Every increasing function* $p(x)$ *satisfies*

$$C_0 p(x_1)\leq p(x_2),\ \ x_1\leq x_2\leq x_1+h_0 p^{1/2}(x_1),\ \ x_1, x_2\in(0,\infty),\ \ C_0>0$$

3. The main Theorem

Theorem. *Let the coefficients of the Sturm-Liouville differential expression*

$$a[\cdot]=-\frac{d}{dx}\,p(x)\frac{d}{dx}+q(x),\ \ 0<x\leqslant 1.$$

satisfy the following conditions.

i) $p(x)\in W_2^1(X,1)$ *and* $q(x)\in L_2(X,1)$ *for all* X, $0<X<1$.

ii) There is a constant $C_p>0$ *such that*

$$0<p(x)\leq C_p x^2,\ \ q(x)\geq 0,\ \ 0<x\leq 1.$$

iii) There exist positive constants h_0, c_0, C_0 *such that*

$$c_0 p(x_1)\leq p(x_2)\leq c_0 p(x_1),\ \ x_1\leq x_2\leq x_1+h_0 p^{1/2}(x_1),\ \ 0<x_i\leqslant 1,\ \ i=1,2.$$

Then the spectrum of any self-adjoint extension of A_0, $A_0 u=a[u]$, $D(A_0)=C_0^{\infty}(0,1)$ *is discrete if and only if*

$$\lim_{x\to 0}\frac{1}{p^{1/2}(x)}\int_x^{x+hp^{1/2}(x)} q(t)dt=\infty \ \ for\ each\ h,\ 0<h\ min(h_0,\ C_p^{-(1/2)}).$$

Proof. we define a sequence x_v by

$$x_{v+1}=x_v-hp^{1/2}(x_v),\ \ 0<h<min\ (h_0,\ C_p^{-1/2}),\ \ v=1,2,\ldots,\ \ x_1=1,$$

and we have $x_v\to 0$ as $v\to\infty$.

For a given $\varepsilon>0$ we find a value x_ε, $0<x_\varepsilon$ such that

$$(A_0 u, u)_{(0,1)}\geq(2\varepsilon)^{-1}\ \|u\|^2_{(0,1)},\ \ u\in C_0^{\infty}(0,x_\varepsilon)$$

Therefere $\sigma_e(\hat{A})=\phi$.

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