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Almost Continuous Mappings on an Almost Regular Spaces

By Chong-Moon Shin and Hyung-Jae Lee
Kyeongju College, Dongkuk University, Kyeongju, Korea

I. Introduction

The object of this paper is to introduce an almost continous mappings on an almost regular spaces. In general, almost continuous mappings are weaker than continuous mappings and stronger than weakly continuous mappings. So, we studied the conditions of the mapping under which weakly continuous mappings are almost continuous mappings and the image of almost regular space is almost regular. We do not know whether every θ-continuous mapping is almost continuous or not. In this paper, we found that every θ-continuous mapping is almost continuous if Y is almost regular space. But this map can not be continuous. (See example 3.9.) Vincent J. Mancuso [1] introduced a concept of almost locally connected space and studied the equivalent conditions of almost locally connected space. In the last chapter of this paper, we studied the conditions of the mapping under which the image of almost locally connected space is almost locally connected space.

II. Preliminaries and Notations

In this paper, (X, τ) will denote a topological space X with topology τ and all maps are onto.

Definition 2.1. A subset U of a topological space X is said to be ***regular open***(or regular closed) if $U=\text{int}\ (\text{cl}U)$ (or $U=\text{cl}(\text{int}\ U)$).

Clearly, a set is regular open iff its complement is regular closed.

Definition 2.2. A space (X, τ) is said to be ***almost regular*** if for each x in X and each neighborhood U of x, there is a regular open neighborhood V of x such that $\text{cl}V \subset \text{int}\ (\text{cl}U)$.

Remark 2.3. Every regular space is almost regular. But the converse is not true. (See example 3.4. [1])

Theorem 2.4. *(X, τ) is almost regular iff for each x in X and each regular open neighborhood U of x, there is a regular open nbd V of x such that $x \in V \subset \bar{V} \subset U$.* (*Theorem* 2.2. [5])

Definition 2.5. A space (X, τ) is said to be ***semi-regular*** if X has a basis consisting of regular open set.

Definition 2.6. A mapping $f: X \to Y$ is said to be ***almost continuous at a point x in X*** if for every nbd V of $f(x)$, there is a nbd U of x such that $f(U) \subset \text{int}\ (\text{cl}V)$. f is ***almost continuous*** on X if f is almost continuous at each point of X.

Theorem 2.7. ***f is almost continuous iff inverse image of every regular open subset of Y is open subset of X.*** (*Theorem* 2.2. [6])

Definition 2.8. A mapping $f: X \to Y$ is said to be *weakly continuous* if for each point x in X and each nbd V of $f(x)$, there is a nbd U of x such that $f(U) \subset \bar{V}$.

Definition 2.9. A mapping $f: X \to Y$ is said to be *θ-continuous* if for each point x in X and each nbd V of $f(x)$, there is a nbd U of x such that $f(\bar{U}) \subset \bar{V}$.

Remark 2.10. Every almost continuous mapping is weakly continuous. But the converse is not true. (See example 2.3. [6]). And every θ-continuous mapping is weakly continuous. But θ-continuous mapping may fail to be almost continous. We do not know, however, whether every almost continous mapping is θ-continous or not.

Definition 2.11. A mapping $f: X \to Y$ is said to be *almost-open* (almost-closed) if the image of every regular open (regular closed) subset of X is open (closed) subset of Y.

Clearly, a 1-1 mapping is almost open iff it is almost closed.

Definition 2.12. A space is called *Urysohn space* if for every pair of distinct points x and y, there is a nbd U and V such that $x \in U$, $y \in V$ and $\bar{U} \cap \bar{V} = \phi$.

Theorem 2.13. *Every almost regular Hausdorff space is Urysohn space. (See* [5]).

Definition 2.14. A space (X, τ) is said to be *almost compact* (or nearly compact) if each open cover has a finite subcover whose closures (or interiors of the closures) cover the space.

Theorem 2.15. *Let $f: X \to Y$ be an almost continuous mapping. If X is almost compact, then Y is almost compact. (See* [6])

III. Almost continuous mappings on an almost regular spaces

Lemma 3.1. *Let $f: X \to Y$ be an open, almost continuous mapping. Then for each open V of Y, $cl(f^{-1}(V)) = f^{-1}(cl(V))$.*

Proof. See the corollary of Theorem 7. [2]

Theorem 3.2. *Let $f: X \to Y$ be an open almost continuous,* 1-1 *mapping. If X is almost regular, then Y is almost regular.*

Proof. Let M be a regular open nbd of $f(x)$. Since f is almost continuous, $f^{-1}(M)$ is an open nbd of x in X. Since X is almost regular, there is a regular open nbd V of x such that

$$x \in V \subset \bar{V} \subset \text{int } (\text{cl } f^{-1}(M)).$$

Since f is open and 1-1, $f(x) \in f(V) \subset \overline{f(\bar{V})} \subset f \text{ (int (cl } f^{-1}(M)))$

$$\subset \text{int } f(\text{cl } f^{-1}(M))$$

By Lemma 3.1., $f(x) \in \overline{f(\bar{V})} \subset f(V) \subset \text{int cl}(M) = M$.

Since $f(V)$ is open, int (cl $f(V)$) is regular open nbd of x.

Hence $f(x) \in f(V) \subset \text{int cl}(f(V)) \subset \text{cl } f(V) \subset M$. Therefore Y is almost regular.

Corollary 3.3. *Let $f: X \to Y$ be an open almost continuous* 1-1 *mapping. If X is almost regular and Y is Hausdorff space, then Y is Urysohn space.*

Proof. By Theorem 3.2., Y is almost regular. By Theorem 2.13. Y is Urysohn space.

Theorem 3.4. *Let $f: X \to Y$ be a weakly continuous, almost open mapping. If X is semi-regular, then f is almost continuous.*

Proof. Let V be an open nbd of $f(x)$. Since f is weakly continuous, there is an open nbd U of x such that $f(U) \subset \bar{V}$. Since X is semi-regular, there is a regular open nbd W of x such that $x \in$

$W \subset U$. Since f is almost open, $f(x) \in f(W) = f(W)^0 \subset$ int cl(V). Hence f is almost continuous.

Theorem 3.5. *Let $f: X \to Y$ be an almost open, weakly continuous mapping. If X and Y are semi-regular, then f is continuous.*

Proof. Let V be an open nbd of $f(x)$. Since Y is semi-regular, there is a regular open nbd W of $f(x)$ such that $f(x) \in W \subset V$ By Theorem 3.4., f is almost continuous. Hence there is an open nbd U of x such that $f(x) \in f(U) \subset \text{int cl}(W) = W \subset V$. Hence f is continuous.

Theorem 3.6. *If f is a weakly continuous mapping of a space X onto an almost regular space Y, then f is almost continuous.*

Proof. Let M be a regular open neighborhood of $f(x)$. Claim: $f^{-1}(M)$ is open neighborhood of x. Let $x \in f^{-1}(M)$. Then $f(x) \in M$. Since Y is almost regular, there is a regular open nbd V of $f(x)$ such that $f(x) \in V \subset \bar{V} \subset M$. Since f is weakly continuous, there is an open nbd U of x such that $f(x) \in f(U) \subset \bar{V} \subset M$. Hence $x \in U \subset f^{-1}(M)$. Therefore f is almost continuous.

Theorem 3.7. *If f is an open continuous mapping of X onto Y and if g is a mapping of Y into Z, then $g \circ f$ is weakly continuous iff g is weakly continuous.*

Proof. ($\Rightarrow$) Let W be an open nbd of $g(f(x))$. Since $g \circ f$ is weakly continuous, there is an open nbd U of x such that

$$g \circ f(U) = g(f(U)) \subset \bar{W}$$

Since f is open, $f(U)$ is open nbd of $f(x)$. Hence g is weakly continuous.

($\Leftarrow$) Let W be an open nbd of $g \circ f(x)$. Since g is weakly continuous, there is an open nbd V of $f(x)$ such $g(V) \subset \bar{W}$.

Since f is continuous, there is an open nbd U of x such that $f(U) \subset V$. Hence $g \circ f(U) \subset g(V) \subset \bar{W}$. Therefore $g \circ f$ is weakly continuous.

Theorem 3.8. *Let $f: X \to Y$ be a θ-continuous mapping. If Y is almost regular, then f is almost continuous.*

Proof. Let M be a regular open nbd of $f(x)$. Since Y is almost regular, there is a regular open nbd V of $f(x)$ such that

$$f(x) \in V \subset \bar{V} \subset M$$

Since f is θ-continuous, there is a nbd U of x such that

$$f(x) \in f(U) \subset f(\bar{U}) \subset \bar{V} \subset M$$

Hence f is almost continuous.

But f can not be continuous mapping in Theorem 3.8., The following is the example.

Example 3.9. Let R be the set of real numbers and let $\mathfrak{U}$ be the usual topology on R and let τ be the another topology on R generated by the union of $\mathfrak{U}$ and τ', the topology of countable complement on R. Then (R, τ) is an almost regular space.

Define $i: (R, \mathfrak{U}) \to (R, \tau)$ be the identity mapping.

Then i is θ-contnuous. Hence i is almost continuous. But i is not continuous at any point.

Lemma 3.10. *The complement of a regular closed set is regular open. (See [3])*

Theorem 3.11. *Regular closed subset of a nearly compact space is nearly compact.*

Proof. Let X be a nearly compact and let A be a regular closed subset of X. Let $\{U_i\}$ be an open cover of A.

Then $\{U_i\} \cup (X-A)$ is open cover of X. Since X is nearly compact, there is a finite subcover such that $X \subset \bigcup_{i=1}^{n} \bar{U}_i^0 \cup (\overline{X-A})^0$ Since A is regular closed, by Lemma 3.11., $X-A$ is regular open. Hence $X-A=\overline{X-A}^0$

$$A = A \cap X \subset (A \cap (\bigcup_{i=1}^{n} \bar{U}_i^0)) \cup (A \cap (\overline{X-A})^0) \subset \bigcup_{i=1}^{n} \bar{U}_i^0$$

Therefore A is nearly compact.

Theorem 3.12. *Let $f: X \longrightarrow Y$ be an almost continuous,* 1-1 *mapping. If X is nearly compact and Y is Urysohn space, then f is almost open.*

Proof. Let U be a regular open set in X. Then $X-U$ is regular closed. By Theorem 3.11., $X-U$ is nearly compact. By Theorem 2.15., $f(X-U)$ is almost compact. Since Y is Urysohn space and f is 1-1, $f(X-U)=Y-f(U)$ is closed. Hence $f(U)$ is open.

IV. Almost locally connected spaces

Definition 4.1. A space X is *almost locally connected at a point p* in X if given a regular open nbd U of p, there is a connected nbd V of p such that $V \subset U$. X is almost locally connected provided X is almost locally connected at each of its points.

Remark 4.2. Every locally connected space is almost locally connected space. But the converse is not true. (See Ex. 3.4. [1])

Definition 4.3. A map $f: X \rightarrow Y$ is *connected* if $f(C)$ is connected whenever C is connected in X.

Theorem 4.4. *Let $f: X \rightarrow Y$ be an almost continuous, open, connected mapping. If X is almost locally connected, then Y is almost locally connected.*

Proof. Let G be a regular open nbd of $f(x)$. Since f is almost continuous, $f^{-1}(G)$ is open neighborhood of x. Hence int cl $f^{-1}(G)$ is regular open neighborhood of x. Since X is almost locally connected, there is a connected set V such that $x \in V \subset$ int cl $f^{-1}(G)$. By Lemma 3.1., $f(x) \in f(V) \subset f$ (int cl $f^{-1}(G)) \subset$ int cl $G=G$. Since f is open and connected, Y is almost locally connected.

Theorem 4.5. *Let $f: X \rightarrow Y$ be an almost open, continuous mapping. If X is almost locally connected space, then Y is almost locally connected space.*

Proof. Let G be a regular open neighborhood of $f(x)$. Since f is continuous, $f^{-1}(G)$ is open nbd of x. Hence int cl $f^{-1}(G)$ is a regular open nbd of x. Since X is almost locally connected, there is a connected nbd V of x such that $x \in V \subset$ int cl $f(G)$. Since V is connected and $V \subset$ int cl$V \subset$ cl$V$, int cl$V$ is connected regular open nbd of x. Hence $x \in V \subset$ int cl$V \subset$ int cl $f^{-1}(G)$.

$$f(x) \in f(\text{int cl}V) \subset f(\text{int cl } f^{-1}(G)) \subset \text{int cl}G = G.$$

Since f is almost open and continuous, f(int cl V) is connected nbd of x. Hence Y is almost locally connected space.

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