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On Bounded Linear Mappings on Topological Vector Spaces

By Lee Hyung-Jae
Kyungju College, Dongguk University, Kyungju, Korea.

I. Introduction

Just as the notion of a metric space generalizes to that of a topological space, so the notion of a normed vector space generalizes to that of a topological vector space (briefly, TVS).

It is well known that a linear mapping from a normed vector space into a normed vector space is continuous if and only if it is bounded [1, P. 210].

In this paper, we show that a similar property holds for linear mappings on TVS. Since the topology of a TVS is determined by the origin θ, in section II, we study the properties of neighborhoods of θ. In section III, we define the bounded sets of TVS and the bounded linear mappings. In section IV, we generalize the boundedness and continuity for linear mappings on a normed vector space to the case on TVS.

II. Definitions and Preliminaries

Definition 2.1. A *topological vector space* (E, τ) over $\mathbf{C}$ is a vector space E over $\mathbf{C}$, equipped with a topology τ that the mappings $(x, y) \to x+y$ from $E \times E$ into E and $(\alpha, x) \to \alpha x$ from $\mathbf{C} \times E$ into E are continuous. Throughout this paper, $\mathbf{C}$ denotes the field of complex numbers.

TVS has the algebraic structure as a vector space and the topological structure as a topological space.

Let E be a TVS. Then the mapping $x \to x+x$ is a homeomorphism of E onto itself, and the mapping $x \to \alpha x (\alpha \neq 0)$ is a topological automorphism of E. Since the mapping $x \to x+x$ is contiuous and the mapping $x \to \alpha x (\alpha \neq 0)$ is linear and continuous and the image of these mappings is the whole of E, the inverse mappings $x \to x-x$ and $x \to \frac{1}{\alpha} x$ exist and have the same properties.

Therefore, if U is a neighborhood of θ, $u+x_0$ is a neighborhood of x_0. Further, if U is a neighborhood of θ, so is αu, for $\alpha \neq 0$. Hence the topology of a TVS is completely determined by a filter of neighborhoods of θ.

The following theorem is a criterion, expressed in terms of the neighborhoods of θ, for TVS.

Theorem 2.1. *A filter $\mathcal{F}$ on a vector space E is the filter of neighborhoods of the origin in a topology compatible with the linear structure of E if and only if it has the following properties:*

(1) *The origin belongs to every subset U belonging to $\mathcal{F}$.*

(2) *To every $U \in \mathcal{F}$ there is $V \in \mathcal{F}$ such that $V+V \subset U$.*

(3) *For every $U \in \mathcal{F}$ and for every $\alpha \in \mathbf{C}$, $\alpha \neq 0$,* we have $\alpha U \in \mathcal{F}$.

(4) *Every* $U \in \mathcal{F}$ *is absorbing.*

(5) *Every* $U \in \mathcal{F}$ *contains some* $V \in \mathcal{F}$ *which is balanced.*

Proof. The proof can be found in [2, P. 22].

Definition 2.2. A subset A of a vector space E is said to be *absorbing* if to every $x \in E$ there is a number $C_x > 0$ such that, for all $\alpha \in \mathbf{C}$, $|\alpha| \leq C_x$, we have $\alpha x \in A$.

Definition 2.3. A subset B of a vector space E is said to be *balanced* if for every $x \in B$ and every $\alpha \in \mathbf{C}$, $|\alpha| \leq 1$, we have $\alpha x \in B$.

Definition 2.4. The metric d on a vector space E is said to be *translation invariant* if the following condition is verified:

$$d(x, y) = d(x+z, \ y+z) \text{ for all } x, y, z \in E.$$

Definition 2.5. A TVS is said to be *metrizable* if the topology of the TVS is given by a translation invariant metric.

The following theorem is needed for the theorem 4.2.

Theorem 2.2. Let (E, τ) be a metrizable TVS. Then there is a countable basis $\{U_n | n=1, 2, 3, \ldots\}$ of neighborhoods of θ in E such that each U_n is balanced $U_1 \supset U_2 \supset U_3 \supset \ldots$ is totally ordered.

Proof. Since E is a metrizable TVS, there is a metric d: $E \times E \to \boldsymbol{R}$ defining τ. For each $n \in \boldsymbol{N}$, if we set

$$W_n = \{x \in E | d(\theta, x) < \frac{1}{n}\},$$

then $\{W_n | n=1, 2, 3, \ldots\}$ is a countable basis of neighborhoods of θ, and so $\bigcap_{n=1}^{\infty} W_n = \{\theta\}$.
By Theorem 2.1. (5), each W_i contains a balanced neighborhood V_i of θ. If we take

$$U_1 = V_1, U_2 = V_1 \cap V_2, \ldots, U_n = V_1 \cap V_2 \cap \ldots \cap V_n, \ldots$$

as a basis of neighborhoods of θ, then U_n is balanced and

$$U_1 \supset U_2 \supset U_3 \supset \ldots.$$

Furthermore, since $U_n \subset W_n$ for $n \in \boldsymbol{N}$, we have

$$\bigcap_{n=1}^{\infty} U_n \subset \bigcap_{n=1}^{\infty} W_n = \{\theta\}$$

and hence $\bigcup_{n=1}^{\infty} U_n = \{\theta\}$.

III. Bounded sets

Definition 3.1. A subset B of the TVS E is said to be *bounded* if to every neighborhood U of the origin θ in E there is a number $\lambda \geq 0$ such that

$$B \subset \lambda U.$$

Since a normed vector space is a TVS, we can define the bounded sets of the normed vector space in this way.
A subset B of the normed vector space $(E, \| \ \|)$ is bounded if there is a $\lambda \geq 0$ such that

$$B \subset \{x \in E | \|x\| < \lambda\}$$

The following properties are obvious.

(1) Every subset of a bounded set is bounded.

(2) Finite unions of bounded sets are bounded.

Theorem 3.1. *In a TVS, compact sets are bounded.*

Proof. The proof can be found in [2, P. 137].

Theorem *If E is Hausdorff, then a converging sequence $\{x_n\}$ in E is bounded.*

Proof. Let $\{x_n\}$ be a sequence converging to x_0. Then the set

$$K=\{x_n \mid n=1, 2, 3, \ldots\} \cup \{x_0\}$$

is compact. By Theorem 3.1., K is bounded.

Since a subset of the bounded set is bounded, $\{x_n\}$ is bounded.

Theorem 3.3. *In a TVS E, a subset B of E is bounded if and only if every sequence con tained in B is bounded in E.*

Proof. If B is bounded, then every sequence contained in B is obviously bounded.

Conversely, suppose that B is unbounded. Then there is a neighborhood U of θ in E such that

$$B \not\subset nU, \quad \text{for } n=1, 2, 3, \ldots$$

Hence, for each $n=1, 2, 3, \ldots$, there is a $x_n \in B-nU$ and hence the sequence $\{x_n\}$ cannot be bounded.

Theorem 3.4. *The image of a bounded set B under a continuous linear mapping T of a TVS E into a TVS F is bounded.*

Proof. Since T is continuous, given a neighborhood V of θ in F, there is a neighborhood U of θ in E such that

$$T(U) \subset V.$$

Since B is bounded, it follows from $B \subset \lambda U (\lambda \geq 0)$ that $T(B) \subset T(\lambda U) = \lambda T(U) \subset \lambda V$. Hence $T(B)$ is bounded in F.

IV. Bounded linear mappings

Definition 4.1. Let E and F be TVS. A linear mapping $T: E \rightarrow F$ is said to be bounded if $T(B)$ is a bounded subset of F for every bounded set $B \subset E$.

Note that, by Theorem 3.4., a continuous linear mapping from a TVS into another TVS is bounded.

Theorem 4.1. *Let E and F be normed vector spaces. Then a linear mapping $T: E \rightarrow F$ is bounded if and only if T is continuous.*

Proof. The proof can be found in [1, P. 210].

Theorem 4.1. is generalized in the following theorem for the linear mappings on TVS.

Theorem 4.2. *Let E be a metrizable TVS and let F be a TVS. Then a linear mapping $T: E \rightarrow F$ is bounded if and only if T is continuous.*

Proof. Since, by Theorem 3.4., the sufficiency of the condition is obvious, we prove its necessity.

Suppose that T is not continuous. Then there is a neighborhood V of θ in F whose preimage $T^{-1}(V)$ is not a neighborhood of θ in E.

By Theorem 2.2., there is a countable basis $\{U_n | n=1,2,3,\ldots\}$ of neighborhoods of θ in E such that each U_n is balanced and $U_1 \supset U_2 \supset U_3 \cdots$ is totally ordered.

Since $T^{-1}(V)$ is not a neighborhood of θ, for all $n \in N$, we have

$$\frac{1}{n} U_n \not\subset T^{-1}(V).$$

Hence there is $x_n \in \frac{1}{n} V_n$ such that $x_n \notin T^{-1}(V)$. Since $nx_n \in U_n$, the sequence $\{nx_n\}$ converges to θ in E. By Theorem 3.2., $\{nx_n\}$ is bounded in E. Since T is bounded, the sequence $\{nT(x_n)\}$ is bounded in F. Hence there is a $\lambda \geq 0$ such that

$$nT(x_n) \in \lambda V \text{ for all } n \in N.$$

Since V is balanced, we have

$$T(x_n) \in \frac{\lambda}{n} V \subset V \text{ for all } n \geq \lambda.$$

This contradicts our assumption; $x_n \notin T^{-1}(V)$.
Therefore T is continuous.

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