韓國數學教育學會誌<數學教育>
1983. 12. Vol. XXII, No. 1

Journal of the Korea Society of Mathematical Education
Dec. 1983. Vol. XXII, No. 1

Study on the Lebesgue-Stieltjes Integral of the Complex Valued Function on $[a, b]$

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1. Introduction

FIRST; I considered of the Lebesgue-Stieltjes integral of the real valued functions on $[a, b]$ and investigated a few the properties of the Lebesgue-Stieltjes integral on $[a, b]$.

SECOND; I considered of the Lebesgue-Stieltjes integral of the complex-valued functions on $[a, b]$.

THIRD; I investigated a few the properties and applications of the Lebesgue-Stieltjes integral of the complex valued functions on $[a, b]$.

2. The main results.

Definition 1. Let g be a nondecreasing function on $[a, b]$. Suppose that $f[a, b]$ belongs to $\mathcal{L}^g$. The *Lebesgue-Stieltjes integral* of f over $[a, b]$ is defined to be;

$$\int_{[a,b]} f dg = \int f x[a, b] dg.$$

Theorem 1. *Let $\phi=\phi_1+\phi_2$ be increasing function on $[a, b]$, such that $\phi(x)=\phi_1(x)+\phi_2(x)$ and $f=f_1+f_2$ positive function on $[a, b]$ such that $f(x)=f_1(x)+f_2(x)$.*

Let $\int f(x)d\phi(x)$ be lim $\sum f(\xi_i)\{\phi(x_i)-\phi(x_{i-1})\}$.

Suppose that if $\int_a^b fd\phi$ or $\int_a^b \phi df$ exists, then the other exists.

Then $\int_a^b fd\phi+\int_a^b \phi df=\sum_{j=1}^{2} f_j(b)\phi_j(b)-\sum_{j=1}^{2} f_j(a)\phi_j(a)+\int_a^b f_1 d(\phi_2)+\int_a^b \phi_1 d(f_2)+\int_a^b f_2 d(\phi_1)+\int_a^b \phi_2 d(f_1)$.

Proof. Suppose that $\int_a^b (f_1+f_2)d(\phi_1+\phi_2)$ exists. Let $a=x_0\le\xi_1\le x_2\le\xi_2\le x_3\le\ldots\le x_{n-1}\le\xi_n\le x_n=b$ be any partition of $[a, b]$. Let $\xi_0=a$, $\xi_{n+1}=b$, $T_0=\sum_{i=1}^{n}(f_1+f_2)(x_{i-1})\{(\phi_1+\phi_2)(\xi_i)-(\phi_1+\phi_2)(\xi_{i-1})\}$, and $T_1=\sum_{i=1}^{n}(\phi_1+\phi_2)(\xi_i)\{(f_1+f_2)(x_i)-(f_1+f_2)(x_{i-1})\}$.

Hence $T_0+T_1=\sum_{j=1}^{2} f_j(b)\phi_j(b)-\sum_{j=1}^{2} f_j(a)\phi_j(a)+\int_a^b f_1 d(\phi_2)+\int_a^b f_2 d(\phi_1)+\int_a^b \phi_1 d(f_2)+\int_a^b \phi_2 d(f_1)$.

If either $\max(x_i-x_{i-1})$ or $(\max\xi_i-\xi_{i-1})$ tends to 0, so does the other. Since $\int_a^b (f_1+f_2)d(\phi_1+\phi_2)$ exists, it is the limit of T as $\max(x_i-x_{i-1})$ tends to 0. That is; $\int_a^b (f_1+f_2)d(\phi_1+\phi_2)=\int_a^b f_1 d(\phi_1)+\int_a^b f_2 d(\phi_2)+\int_a^b f_1 d(\phi_2)+\int_a^b f_2 d(\phi_1)$.

Since $\max(x_i-x_{i-1})\to 0$, $\max(\xi_i-\xi_{i-1})\to 0$, T_1 tends to a limit, and so $\int_a^b(\phi_1+\phi_2)d(f_1+f_2)$ exists.

That is; $\int_a^b(\phi_1+\phi_2)d(f_1+f_2)=\int_a^b\phi_1 d(f_1)+\int_a^b\phi_2 d(f_2)+\int_a^b\phi_1 d(f_2)+\int_a^b\phi_3 d(f_1)$.

Hence $\int_a^b(f_1+f_2)d(\phi_1+\phi_2)+\int_a^b(\phi_1+\phi_2)d(f_1+f_2)=\sum_{j=1}^2 f_j(b)\phi_j(b)-\sum_{j=1}^2 f_j(a)\phi_j(a)+\int_a^b f_1 d(\phi_2)+\int_a^b f_2 d(\phi_1)+\int_a^b f_2 d(\phi_1)+\int_a^b\phi_1 d(f_2)+\int_a^b\phi_2 d(f_1)$.

Definition 2. Let function g be real valued and nondecreasing.

If $f=f_1+if_2$ is a complex-valued function defined on $[a,b]$ and if $\int f_1\ dg$ and $\int f_2\ dg$ exist, then we say that f is *Lebesque-Stieltjes integrable* with respect to g. and $\int_a^b fdg=\int_a^b f_1 dg+i\int_a^b f_2 dg$.

Theorem 2. *Let $\phi=\phi_1+\phi_2$ be real valued nondecreasing function on $[a,b]$.*

If $f=f_1+if_2$ is a complex-Valued function defined on $[a,b]$ and if $\int_a^b f_1\,d\phi_1$, $\int_a^b f_2\,d\phi_2$, $\int_a^b f_1 d\phi_2$ and $\int_a^b f_2 d\phi_1$ exist, then $\int_a^b(f_1+if_2)d(\phi_1+\phi_2)=\int_a^b f_1 d\phi_1+\int_a^b f_1 d\phi_2+i\left(\int_a^b f_2 d\phi_1+\int_a^b f_2 d\phi_2\right)$.

Proof. Since $fd(\phi_1+\phi_2)=fd\phi_1+fd\phi_2$, $\int_a^b(f_1+if_2)d(\phi_1+\phi_2)=\int_a^b(f_1+if_2)d\phi_1+\int_a^b(f_1+if_2)d\phi_2=\int_a^b f_1 d\phi_1+i\int_a^b f_2 d\phi_1+\int_a^b f_1 d\phi_2+i\int_a^b f_2 d\phi_2=\int_a^b f_1 d\phi_1+\int_a^b f_1 d\phi_2+i\int_a^b f_2 d\phi_1+\int_a^b f_2 d\phi_2$.

Theorem 3. *If ϕ is nondecreasing and absolutely continuous function on $[a,b]$. Let f_1 and f_2 be Lebesgue-Stieltjes integrable with respect to and $f=f_1+if_2$ complex valued function on $[a,b]$. Then $\int_a^b fd\phi=\int_a^b(f_1+if_2)d\phi=\int_a^b(f_1+if_2)\phi'$.*

Proof. Suppose that f_j is admissible step functions with representation $f_j(x)=0$ for $x\notin[a,b]$ and $f_j(x)=C_k{}^j$ for $x\in(a_{k-1},a_k)$, $j=1,2$, $k=1,2,\dots,n$.

Then $f_j\phi$ is Lebesgue integrable function on $[a,b]$.

$$\int_a^b(f_1+if_2)\phi'=\int_a^b f_1\phi'+i\int_a^b f_2\phi'=\sum_{k=1}^n C^1{}_k\int_{a_{k-1}}^{a_k}\phi'+i\sum_{k=1}^n C_k{}^2\int_{a_{k-1}}^{a_k}\phi'.$$

Since ϕ is the absolute continuous function on $[a,b]$, $\int_a^b(f_1+if_2)d\phi=\sum_{k=1}^n(C_k{}^1+iC_k{}^2)\{\phi(a_k)$

$-\phi(a_{k-1})\}=\sum_{k=1}^n C_k{}^1\{\phi(a_k)-\phi(a_{k-1})\}+i\sum_{k=1}^n C_k{}^2\{\phi(a_k)-\phi(a_{k-1})\}=\sum_{k=1}^n C_k{}^1\int_{a_{k-1}}^{a_k}\phi'+i\sum_{k=1}^n C_k{}^2\int_{a_{k-1}}^{a_q}\phi'$.

Hence by ①, ②, $\int_a^b(f_1+if_2)\phi'=\int_a^b(f_1+if_2)d\phi$.

Theorem 4. *Let $f=f_1+if_2$ and $g=g_1+ig_2$ be complex valued function defined on $[a,b]$ and ϕ be real valued-nondecreasing function on $[a,b]$. Suppose that f_1 and f_2, g_1 and g_2 are Lebesgue-Stieltjes integrable on $[a,b]$. Then $\int_a^b(f+g)d\phi=\int_a^b fd\phi+\int_a^b gd\phi$.*

Proof. Since $f+g=f_1+g_1+i(f_2+g_2)$, $\int_a^b(f+g)d\phi=\int_a^b(f_1+g_1)d\phi+i\int_a^b(f_2+g_2)d\phi$.

Since f_1, f_2 and g_1, g_2 are Lebesgue-Stieltjes integrable on $[a,b]$, $\int_a^b (f_1+g_1)\,d\phi = \int_a^b f_1 d\phi + {}^b{}_a g_1 d\phi$ and $\int_a^b (f_2+g_2)\,d\phi = \int_a^b f_2 d\phi + \int_a^b g_2 d\phi$, Hence $\int_a^b (f+g)\,d\phi = \int_a^b f_1 d\phi + \int_a^b g_1 d\phi + i\int_a^b f_2\,d\phi + i\int_a^b f_2 d\phi + i\int_a^b g_2 d\phi = \int_a^b f_1 d\phi + i\int_a^b f_2 d\phi + \int_a^b g_1\,d\phi + i\int_a^b g_2 d\phi = \int_a^b f d\phi + \int_a^b g d\phi$. Hence $\int_a^b (f+g)\,d\phi = \int_a^b f d\phi + \int_a^b g d\phi$.

Theorem 5. *Let f_1 and f_2 be real valued nonnegative function on $[a,b]$ and $\int_a^b f_1 d\phi + \int_a^b f_2 d\phi$ exist. Let ϕ be nondecreasing function on $[a,b]$ and $f=f_1+if_2$ be complex-valued function on $[a,b]$.*

Then there is $\xi \in (a,b)$ such that $\int_a^b f(x)\,d\phi(x) = f_1(\xi)\sum_{i=1}^{n}\{\phi(x_i)-\phi(x_{i-1})\} + if_2(\xi)\sum_{i=1}^{n}\{\phi(x_i)-\phi(x_{i-1})\}$.

Proof. Since ϕ is nondecreasing function on $[a,b]$, $\int_a^b f(x)\ d\phi(x) = \int_a^b f_1(x)\,d\phi(x) + i\int_a^b f_2 d\phi(x)$.

Since $\int_a^b f(x)\,d\phi(x)$ and $\int_a^b f_2(x)\,d\phi(x)$ are Lebesgue-Stieltjes integral on $[a,b]$. By the first mean value theorem, there is $\xi \in (a,b)$ such that $\int_a^b f(x)\,d\phi(x) = f_1(\xi)\sum_{i=1}^{n}\{\phi(x_i)-\phi(x_{i-1})\} + if_2(\xi)\sum_{i=1}^{n}\{\phi(x_i) -\phi(x_{i-1})\}$.

Theorem 6. *Let f_j be real valued nondecreasing functions on $[a,b]$ $(j=1,2)$ and let $f=f_1+if_2$. Suppose that ϕ is continuous real valued nondecreasing function on $[a,b]$. Let $\mu\phi$ be the Lebesgue-Stieltjes measure corresponding to ϕ.*

Then there exists $\xi \in (a,b)$ such that $\int_a^b f(x)\,d\mu\phi(x) = f_1(a)[\phi(\xi)-\phi(a)] + f_1(b)[\phi(b)-\phi(\xi)] + i\{f_2(a)\ [\phi(\xi)-\phi(a)] + f_2(b)[\phi(b)-\phi(\xi)]\}$.

Proof. Let $f_j(x)=f(a)$ and $\phi(x)=\phi(a)$ for all $x<a$, and let $f_j(x)=f_j(b)$ for all $x>b$ $(j=1,2)$. Let μf_j and $\mu\phi$ be the Lebesgue-Stieltjes measures corresponding to f_j and ϕ. Then $f_j(a-)=f_j(a)$ and $f_j(b+)=f_j(b)$ and $\phi(x+)=\phi(x-)=\phi(x)$ for all $x\in[a,b]$.

$$\int_a^b \phi(x)\,d\mu f_j(x) = \int_a^b \frac{f_j(x+)+f_j(x-)}{2}\,d\mu\phi(x) = f_j(b)\phi(b) - f_j(a)\phi(a).$$

By theorem 5, there exists $\xi \in (a,b)$ such that $\int_a^b \phi(x)\,d\mu f_j(x) = \phi(\xi)\,\mu f_j([a,b]) = \phi(\xi)[f_j(b)-f_j(a)]$ $(j=1,2)$. Since ϕ is continuous, we have $\mu\phi(\{x\})=0$ for all x and so $\int_a^b \frac{f_j(x+)+f_j(x-)}{2}\,d\mu\phi(x) = \int_a^b f_j(x)\,d\mu\phi(x)$.

Hence $\phi(\xi)[f_j(b)-f_j(a)] + \int_a^b f_j(x)\,d\mu\phi(x) = f_j(b)\phi(b) - f_j(a)\phi(a)$. That is; $\int_a^b f_j(x)\,d\mu\phi(x) = f_j(b)\phi(b) - f_j(a)\phi(a) - \phi(\xi)[f_j(b)-f_j(a)] = f_j(a)[\phi(\xi)-\phi(a)] + f_j(b)[\phi(b)-\phi(\xi)]$, Since $\int_a^b f(x) d\phi(x) = \int_a^b f_1(x)\,d\phi(x) + i\int_a^b f_2(x)\,d\phi(x)$.

Hence $\int_a^b f(x)\,d\mu\phi(x) = f_1(a)[\phi(\xi)-\phi(a)] + f_1(b)[\phi(b)-\phi(\xi)] + i\{f_2(a)[\phi(\xi)-\phi(a)] + f_2(b)[\phi(b) -\phi(\xi)]\}$.

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