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On Collectively Compact Operators into (F)-Spaces

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Let X and Y be locally convex spaces and $[X, Y]$ the set of continuous linear operators from X into Y, equipped with the topology of uniform convergence on compact sets. For a subset $\mathscr{K}\subset[X, Y]$ and a subset $B\subset X$, let $\mathscr{K}(B)$ denote $\{K(b) : K\in\mathscr{K},\ b\in B\}$.

A subset $\mathscr{K}\subset[X, Y]$ is said to be *collectively compact* if there exists a O-neighborhood U in X such that $\mathscr{K}(U)$ is relatively compact.

In [6] it was proved that if $\mathscr{K}\subset[X, Y]$ is collectively compact then $\mathscr{K}(B)$ is relatively compact for every bounded subset $B\subset X$. Under what conditions does the collectively compactness of $\mathscr{K}\subset[X, Y]$ follow from the assumption that $\mathscr{K}(B)$ is relatively compact for every bounded subset $B\subset X$? For a compact operator, Grothendieck [3] proved that this is true if X is quasi-normable and Y a Banach space.

In this paper, for collectively compact operators, we show that it also holds if X is a (DF)-space and Y an (F)-space and that a compact set of compact operators is collectively compact.

Recall that a locally convex space X is *quasi-normable* if and only if for every O-neighborhood U in X there exists a O-neighborhood V in X such that for every $\varepsilon>0$ there is a bounded subset $M\subset X$ with $V\subset M+\varepsilon U$ [3].

Theorem 1. *Let $\mathscr{K}\subset[X, Y]$ be such that $\mathscr{K}(B)$ is relatively compact for every bounded subset $B\subset X$. Then $\mathscr{K}$ is collectively compact if Y is an (F)-space and X a (DF)-space. i.e. if X satisfies the following two conditions* [5]:

(a) *X has a countable fundamental system of bounded subsets.*

(b) *Any bornivorous set in X which is two intersection of a sequence of closed absolutely convex O-neighborhoods in X is a neighborhood in X.*

Before we prove this Theorem, we need the following Lemma.

Lemma 2. *The Theorem holds if Y is an (F)-space and X satisfies the following conditions:*

(i) *X is quasi-normable.*

(ii) *For every sequence of closed absolutely convex O-neighborhoods U_n $(n=1, 2, \ldots\ldots)$ there exists a sequence of positive α_n such that $\bigcap_{n=1}^{\infty}\alpha_n U_n$ is a O-neighborhood in X.*

Proof. Let $(V_n)_{n=1}^{\infty}$ be a O-neighborhood base for Y and $f\in\mathscr{K}$. For each $f(V_n)$ let U_n be a closed absolutely convex O-neighborhood in X such that for every $\varepsilon>0$ there exists a bounded set $M=M_{n,\varepsilon}\subset X$ with $U_n\subset M_{n,\varepsilon}+\varepsilon f^{-1}(V_n)$ $(n=1, 2, \ldots)$. By (11) there exist positive α_n such that $U=\bigcap_{n=1}^{\infty}\alpha_n U_n$

is a O-neighborhood in X. Hence $f(U)\subset\alpha_n f(M_{n,\alpha_n^{-1}})+V_n$ $(n=1, 2, \ldots)$.

Thus $f(U)$ is precompact, and therefore relatively compact, since Y is complete.

Proof of Theorem. It is known [7] that every (DF)-space is quasi-normable and easy to see [3] that for a (DF)-space condition (ii) is satisfied.

Corollary 3. *The Theorem holds if Y is a Banach space and X quasi-normable.*

Proof. V denoting the unit ball of Y and $f\in\mathcal{K}$, let U be a O-neighborhood in X such that for all $\varepsilon>0$ there exists a bounded $M\subset X$ with $U\subset M_\varepsilon+\varepsilon f^{-1}(V)$. Then $f(U)\subset f(M_\varepsilon)+\varepsilon V$.

Theorem 4. *Let $\mathcal{K}\subset[X, Y]$ be a compact set of compact operators. Then $\mathcal{K}$ is collectively compact if Y is an (F)-space and X a (DF)-space.*

Proof. Let B be a bounded O-neighborhood in X. For each $x\in B$, define $f_x : [X, Y]\rightarrow Y$ by $f_x(K)=K(x)$ where $K\in[X, Y]$. Consider the set $\mathcal{F}=\{f_x : x\in B\}$ restricted to the compact space $\mathcal{K}$. Let V be a O-neighborhood in Y. Then the set $W=\{K : K(B)\subset V\}$ is a O-neighborhood.

$$\begin{aligned}\text{Now } \mathcal{F}(W) &= \{f_x(K) : f_x\in\mathcal{F},\ K\in W\}\\ &=\{K(x) : K\in W,\ x\in B\}\\ &=W(B)\subset V.\end{aligned}$$

This proves the equicontinuity of $\mathcal{F}$. Hence, so is the closure of $\mathcal{F}$ in the topology of uniform convergence on compact subsets, or equivalently here the topology of pointwise convergence. Now, for each $K\in\mathcal{K}$, $\overline{\mathcal{F}}(K)=\{f(K) : f\in\overline{\mathcal{F}}\}$ is contained in $\overline{\mathcal{F}(K)}=\overline{\{f_x(K) : f_x\in\mathcal{F}\}}=\overline{\{K(x) : x\in B\}}$, which is compact [6]. Therefore $\overline{\mathcal{F}}(K)$ is relatively compact for each $K\in\mathcal{K}$. Since $\mathcal{K}$ is compact, Problem 8-H [4] and the above imply that $\overline{\mathcal{F}}$ is compact in the topology of uniform convergence on compact subsets. Define $\phi : \overline{\mathcal{F}}\times\mathcal{K}\rightarrow Y$ by $\phi(K, f)=K(f)$. Since ϕ is continuous and $\overline{\mathcal{F}}$ and $\mathcal{K}$ are compact, ϕ has compact image in Y. But the image of ϕ contains $\mathcal{K}(B)$; so $\mathcal{K}$ is collectively compact.

Corollary 5. *Every precompact set of compact operators is collectively compact if Y is quasi-complete and X a (DF)-space.*

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