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Absolute Continuity in Signed Measure

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1. Introduction

In this paper, we define absolute continuity in signed measure using the notions of Hahn decomposition and Jordan decomposition, and investigate some properties of measurable space. We also obtain the following results. An attempt will be made to establish and verify concretely the more general framework in which the discussion of absolute continuity and singularity by signed measure still make sense in abstract measurable space.

2. Preliminary

Definition 1. Let μ be a signed measure on $(X, \mathfrak{M})$ and let (A, B) be a Hahn decomposition of X for μ. Define μ^+, μ^- and $|\mu|$ on $\mathfrak{M}$ by; $\mu^+(E)=\mu(E\cap A)$, $\mu^-(E)=-\mu(E\cap B)$ and $|\mu(E)|=\mu^+(E)+\mu^-(E)<+\infty$ for all $E\in\mathfrak{M}$. The set function μ^+, μ^- and $|\mu|$ are called the *positive variation* of μ, the *negative variation* of μ, and the *total variation* of μ, respectively.

Definition 2. An arbitrary measure $\nu:\mathfrak{M}\to R^*$ is said to be *absolutely continuous with respect to* μ, in symbols, $\nu\ll\mu$, if and only if, for every $E\in\mathfrak{M}$, $\mu(E)=0$ implies $\nu(E)=0$. If ν is defined by a nonnegative measurable function $f;X\to R^*$ and μ, then it follows from $\nu(E)=0$ that $\nu\ll\mu$. In case μ is σ-finite, then the converse is also true.

Proposition 1. *If μ is a signed measure and $\{E_n\}$ a disjoint sequence of measurable sets such that $|\mu(\bigcup_{n=1}^{\infty} E_n)|<+\infty$, then the series $\sum_{n=1}^{\infty}|\mu(E_n)|<+\infty$.*

Proposition 2. *Let $(X, \mathfrak{M})$ be a measurable space, let μ be a signed measure on $(X, \mathfrak{M})$ and define functions μ^+ and μ^- on $\mathfrak{M}$ by $\mu^+(E)=sup\,\{\mu(A);A\in\mathfrak{M},\ A\subset E\}$, $\mu^-(E)=-inf\{\mu(A):A\in\mathfrak{M}.\ A\subset E\}$, then both μ^+ and μ^- are positive measures.*

Proposition 3. *If μ is a signed measure on the measurable set $(X, \mathfrak{M})$, then there exist two disjoint sets A and B whose union is X, such that $\mu^+(B)=0$, $\mu^-(A)=0$.*

Proposition 4. *Let μ be a signed measure on the measurable space $(X, \mathfrak{M})$. Then $\mu(E)=\mu^+(E)-\mu^-(E)$ for every measurable set $E\in\mathfrak{M}$ and μ is also bounded.*

3. Main Theorem

Theorem 1. *Let μ and ν be signed measures on $(X, \mathfrak{M})$ and $\mu(E)$ be the Jordan decomposition of μ, then the following are equivalent;*

(a) $\nu\ll\mu$ (b) $\nu^+\ll\mu,\ \nu^-\ll\mu$

Proof.(Necessity) Let $\nu\ll\mu$. Now, let us consider the Jordan decomposition and Hahn decomposition. $|\nu|(E)=\nu^+(E)+\nu^-(E)$, $\nu^+(E)\geqq 0$, $\nu^-(E)\geqq 0$, and $|\nu|(E)=0$, then $\nu^+(E)=0$, $\nu^-(E)=0$.

(Sufficiency) Let $\nu^+(E) \ll \mu(E)$. $\nu^-(E) \ll \mu(E)$. If $\nu^+(E)=0$, $\nu^-(E)=0$, then $|\nu|(E)=\nu^+(E)+\nu^-(E)$. Thus $\nu \ll \mu$, then $\nu^+ \ll \mu$, $\nu^- \ll \mu$.

Theorem 2. *If μ and ν are total finite measures such that $\nu \ll \mu$ and, is not identically zero, then there exist a positive number ε and a measurable set E such that $\mu(E)>0$ and such that E is a positive set for the signed measure $\nu-\varepsilon\mu$.*

Proof. Let $X=A_n \cup B_n$ $(n=1,2,\ldots)$ be a Hahn decomposition with respect to the signed measure $\nu-\left(\frac{1}{n}\right)\mu$, write $A_0=\bigcup_{n=1}^{\infty} A_n$, $B_0=\bigcap_{n=1}^{\infty} B_n$, then $X=A_0 \cup B_0$. $A_0 \cap B_0=\phi$. And since for every n, we have $B_0 \subseteq, B_n$ $\nu(B_0)-(1/n)\mu(B_0) \leqq 0$. that is, $0 \leqq y(B_0) \leqq (1/n)\mu(B_0)$ $(n=1,2,\ldots)$ and consequently $\nu(B_0)=0$. It follows that $\nu(A_0)>0$ and therefore, by hypothesis $\nu \ll \mu$ that $\mu(A_0)>0$. Hence we must have $\mu(A_n)>0$ $(n=1,2\ldots)$ for at least one value of n; if such a value of n, we write $A=A_n$ and $\varepsilon=1/n$, the requirements of the theorem are all satisfied.

Theorem 3. *Considering any two signed measure μ, ν prove that $\nu \perp \mu$ and $\nu \ll \mu$ imply $\nu=0$; for every $X \in \mathfrak{M}$ such that $\mu(X)=0$.*

Proof. There exists a Hahn decomposition $X=A \cup B$, $A \cap B=\phi$ and $|\nu|(A)=0$, $|\mu|(B)=0$. Let $X \in \mathfrak{M}$ be arbitrary given, $|\nu|(X)=|\nu|(X \cap A)+|\nu|(X \cap B)=|\nu|(X \cap B)$. Therefore $\nu \ll \mu$ and $|\mu|(X \cap B)=0$, we have $|\nu|(X \cap B)=0$ thus $|\nu|(X)=0$.

Theorem 4. *If ν_1, ν_2, μ is a signed measure on the measurable space $(X, \mathfrak{M})$ such that $\nu_1 \perp \mu$, $\nu_2 \perp \mu$* then $(\nu_1+\nu_2) \perp \mu$, $(\nu_1-\nu_2) \perp \mu$.

Proof. Since $X=A_1 \cup B_1$, $A_1 \cap B_1=\phi$, $X=A_2 \cup B_2$, $A_2 \cap B_2=\phi$, $|\nu_1|(A_1)=|\mu|(B_1)=0$ $|\nu_2|(A_2)=|\mu|(B_2)=0$, let $A=A_1 \cap A_2$, $B=B_1 \cap B_2$, then $A \cup B=X$, $A \cap B=\phi$.

Therefore $\nu_1+\nu_2=(\nu_1{}^+ + \nu_2{}^+)-(\nu_1{}^- + \nu_2{}^-)$ clearly since $\nu_1{}^+ + \nu_2{}^+ \geqq 0$, $\nu_1{}^- + \nu_2{}^- \geqq 0$ we have $(\nu_1+\nu_2)^+ = \nu_1{}^+ + \nu_2{}^+$, $(\nu_1+\nu_2)^- = \nu_1{}^- + \nu^-{}_2$. Thus $|\nu_1+\nu_2| \leqq (\nu_1{}^+ + \nu_2{}^+) + (\nu_1{}^- + \nu_2{}^-) = (\nu_1{}^+ + \nu_1{}^-) + (\nu_2{}^+ + \nu_2{}^-) = |\nu_1| + |\nu_2|$. Hence we obtain $0 \leqq |\nu_1+\nu_2|(A) \leqq |\nu_1|(A)+|\nu_2|(A)=|\nu_1|(A_1 \cap A_2)+|\nu_2|(A_1 \cap A_2)=0$, $0 \leqq |\mu|(B) \leqq |\mu|B_1+|\mu|(B_2)=0$, clearly have the desired properties, Similarly we have $(\nu_1-\nu_2) \perp \mu$.

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