韓國數學教育學會誌<數學教育>
1983. 12. Vol. XXII, No. 1

Journal of the Korea Society of Mathematical Education
Dec. 1983. Vol. XXII, No. 1

Ekeland's Fixed Point Theorem in Generalized Metric Spaces

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1. Introduction.

A *generalized metric space* is a pair (X, d) of a nonempty set X and a distance function $d: X\times X\to[0,\infty]$ satisfying

(i) $d(x,y)=0$ iff $x=y$,

(ii) $d(x,y)=d(y,x)$,

(iii) $d(x,z)\leqq d(x,y)+d(y,z)$,

for all x, y, z in X. Such a space X is said to be *complete* if every Cauchy sequence in X converges.

Let X be a generalized metric space and let $CL(X)$ be the set of all nonempty closed subsets of X. For A, B in $CL(X)$, define

$$N_\varepsilon(A)=\{y\in X \mid d(x,y)<\varepsilon \text{ for some } x\in A\}, \text{ for } \varepsilon>0.$$
$$H(A,B)=\inf\{\varepsilon>0 \mid A\subset N_\varepsilon(B) \text{ and } B\subset N_\varepsilon(A)\}.$$
$$D(A,B)=\inf\{d(x,y) \mid x\in A,\ y\in B\}$$
$$\delta(A,B)=\sup\{d(x,y) \mid x\in A,\ y\in B\}$$

Then $(CL(X), H)$ is a generalized metric space and H is called the *Hausdorff metric* on $CL(X)$. Obviously, $D(A,B)\leqq H(A,B)\leqq\delta(A,B)$ for all A, B in $CL(X)$.

In 1976, Caristi proved a fixed point theorem in complete metric spaces which aroused a great deal of interest, because it does not assume the continuity of the mapping under consideration [1]. This also extends the Banach's fixed theorem. In proving this, Caristi used the transfinite induction, but Ekeland [6] proved this theorem more easily by using his variational principle [5].

In this paper, we study the Ekeland's fixed point theorem for single-valued and multi-valued functions in generalized metric spaces and reformulate our main results in [16] in Ekeland's form. These results also extend and unify some Banach type fixed point theorems. In proving this, we also follow the method of Park [15].

2. Main Theorems.

First, we begin with the following theorem of Ekeland.

Theorem 1. (Ekeland [6]) *Let X be a generalized complete metric space and $f: X\to X$ be a selfmap. Suppose there exists a function $\varphi: X\to \mathbf{R}\cup\{+\infty\}\not\equiv\infty$ which is l.s.c. and bounded from below such that*

$$d(x,fx)+\varphi(fx)\leqq\varphi(x), \textit{ for all } x \textit{ in } X.$$

Then f has a fixed point.

Proof. See Ekeland [6].

Theorem 2. *Let X be a generalized comp ete metric space and $f: X \to X$ be a selfmap. Suppose that there is a function $\varphi: X \to \boldsymbol{R} \cup \{+\infty\} \not\equiv \infty$, bounded from below such that*

$$d(x, fx) + \varphi(fx) \leqq \varphi(x) \qquad (*)$$

for all x in X. Then there is an $x \in X$ such that $\{f^n x\}_{n=0}^{\infty}$ converges to some $\xi \in X$. Moreover, if φ is f-orbitally l.s.c. and $f\xi \in \overline{\{f^n x\}}$, then ξ is a fixed point of f.

Proof. Since $\varphi \not\equiv \infty$, there is an $x \in X$ such that $\varphi(x) < \infty$. Then $\varphi(fx) \leqq \varphi(x) < \infty$ by (*). Inductively, we see that $\varphi(f^n x) \leqq \varphi(f^{n-1} x) < \infty$ for $n \geqq 1$. Therefore, $\{\varphi(f^n x)\}_{n=0}^{\infty}$ is a real decreasing sequence, which is bounded from below. So $\{\varphi(f^n x)\}_{n=0}^{\infty}$ is convergent. We have

$$d(f^n x, f^{n+1} x) \leqq \varphi(f^n x) - \varphi(f^{n+1} x)$$
$$d(f^{n+1} x, f^{n+2} x) \leqq \varphi(f^{n+1} x) - \varphi(f^{n+2} x)$$
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$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$d(f^{n+p-1} x, f^{n+p} x) \leqq \varphi(f^{n+p-1} x) - \varphi(f^{n+p} x), \text{ for } n, p \geqq 0.$$

By adding all the above, we can see that

$$d(f^n x, f^{n+p} x) \leqq \varphi(f^n x) - \varphi(f^{n+p} x).$$

Since the righthand-side of this inequality goes to 0 as n and p tend to ∞, so does the lefthand-side. Thus $\{f^n x\}_{n=0}^{\infty}$ is a Cauchy sequence in X and converges to some $\xi \in X$.

Suppose further that φ is f-orbitally l.s.c. and $f\xi \in \overline{\{f^n x\}}$. Then $\varphi(\xi) \leqq \liminf_{n \to \infty} \{f^n x\}$ implies that $\varphi(\xi) = \inf_{v \in \overline{\{f^n x\}}} \varphi(v)$. So $\varphi(\xi) \leqq \varphi(f\xi)$. But by(*), $d(\xi, f\xi) + \varphi(f\xi) \leqq \varphi(\xi)$ and this is possible only when $\xi = f\xi$. This completes the proof.

Remark. If φ is l.s.c. on X, then the condition $f\xi \in \overline{\{f^n x\}}$ is not needed to verify that ξ is a fixed point of f. See Theorem 2 of Ekeland [6].

Example. Let $X = [0, 1] \cup \{2\}$ and $d: X \times X \to [0, \infty]$ be defind by

$$d(x, y) = |x - y|, \text{ if } x \neq 2, \ y \neq 2,$$
$$d(x, y) = \infty, \text{ if } x = 2 \text{ or } y = 2.$$

Then (X, d) is generalized complete metric space. Define $f: X \to X$ and $\varphi: X \to \boldsymbol{R} \cup \{+\infty\}$ as follows;

$$f(x) = \begin{cases} \dfrac{x}{2}, & \text{if } x \neq 0 \\ 2, & \text{if } x = 0 \end{cases}$$

$$\varphi(x) = \begin{cases} \dfrac{1}{1-x}, & \text{if } x \neq 0, 1, 2 \\ \infty, & \text{if } x = 0, 1, \text{ or } 2 \end{cases}$$

Then (*) holds for all x in X, but f has no fixed point. Indeed, $\lim f^n x = 0$ for all $x \in X$, but φ is not f-orbitally l.s.c. at 0.

Theorem 3. *Let X be a generalized metric space and $f: X \to CL(X)$ be a map. Suppose there is a function $\varphi: X \to \boldsymbol{R} \cup \{+\infty\} \not\equiv \infty$, which is bounded from below and such that*

$$^{\forall} x \in X, \ ^{\exists} y_x \in fx, \ d(x, y_x) + \varphi(y_x) \leqq \varphi(x). \qquad (**)$$

Then there is an iterative sequence $\{u_n\}_{n=0}^{\infty}$, $u_n \in f u_{n-1}$, which converges to some $\xi \in X$. Moreover, *if φ is l.s.c. on $\overline{\{u_n\}}$ and $y_\xi \in \overline{\{u_n\}}$, then ξ is a fixed point of f, i.e. $\xi \in f\xi$.*

Proof. Choose u_0 in X so that $\varphi(u_0) < \infty$. Then there is a $u_1 \in fu_0$ such that $d(u_0, u_1) + \varphi(u_1) \leqq \varphi(u_0)$. Hence $\varphi(u_1) \leqq \varphi(u_0) < \infty$. Inductively, we can choose a sequence $\{u_n\}_{0=u}^{\infty}$ such that

$$u_n \in fu_{n-1},$$
$$\varphi(u_n) \leqq \varphi(u_{n-1}), \text{ and}$$
$$d(u_{n-1}, u_n) + \varphi(u_n) \leqq \varphi(u_{n-1})$$

for all $n \geqq 1$. Therefore, as in the proof of theorem 2, we can see that $\{u_n\}$ converges to some $\xi \in X$. Suppose now that φ is l.s.c. on $\overline{\{u_n\}}$, then $\varphi(\xi) = \inf_{x \in \overline{\{u_n\}}} \varphi(x)$ and so $\varphi(\xi) \leqq \varphi(y_\xi)$ if $y_\xi \in \overline{\{u_n\}}$. But this is possible only when $\xi \in f\xi$.

3. Applications.

Let X be a generalized metric space and $f: X \to X$ be a selfmap. Consider the following type of contraction conditions;

(1) $d(fx, fy) \leqq a_1 d(x, y)$, $0 \leqq a_1 \leqq 1$. Diaz and Margolis [4], Jung [11].

(2) $d(fx, fy) \leqq a_1 d(x, y) + a_2 d(x, fx) + a_3 d(y, fy)$, $a_1, a_2, a_3 \geqq 0$ and $a_1 + a_2 + a_3 < 1$. Reich [18].

(3) $d(fx, fy) \leqq a_1 d(x, y) + a_2 d(x, fx) + a_3 d(y, fy) + a_4 [d(x, fy) + d(y, fx)]$, $a_1, a_2, a_3, a_4 \geqq 0$ and $a_1 + a_2 + a_3 + 2a_4 < 1$. Iseki [18].

(4) $d(fx, fy) \leqq a_1 \max \{d(x, y), d(x, fx), d(y, fy), \frac{1}{2}[d(x, fy) + d(y, fx)]\}$, $0 \leqq a_1 \leqq 1$. Ćirić [2].

Clearly, (1), (2) or (3) respectively implies (4). And (4) can be reformulated in our condition in theorem 2. Indeed, if we define $\varphi(x) = \frac{1}{1 - a_1} d(x, fx)$, then φ satisfies (*) and f-orbitally l.s.c. in X.

In multi-valued case, let X be a generalized complete metric space and $f: CL(X) \to X$ be a function. Consider the following conditions;

(1) $H(fx, fy) \leqq a_1 d(x, y)$, $0 \leqq a_1 \leqq 1$. Nadler [14].

(2) $H(fx, fy) \leqq a_1 D(x, fx)$, $0 \leqq a_1 \leqq 1$. Czerwik [3].

(3) $H(fx, fy) \leqq a_1 [D(x, fx) + D(y, fy)]$, $0 \leqq a_1 < \frac{1}{2}$. Kaulgud [12].

(4) $H(fx, fy) \leqq a_1 d(x, y) + a_2 D(x, fx) + a_3 D(y, fy)$, $a_1, a_2, a_3 \geqq 0$ and $a_1 + a_2 + a_3 < 1$. Ray [17], Reich [18].

(5) $D(y, fy) \leqq a_1 d(x, y) + a_2 D(x, fx)$ for all $y \in fx$, $a_1, a_2 \geqq 0$ and $a_1 + a_2 < 1$. Himmelberg [17].

(6) $H(fx, fy) \leqq a_1 d(x, y) + a_2 [D(x, fx) + D(y, fy)] + a_3 [D(x, fy) + D(y, fx)]$, $a_1, a_2, a_3 \geqq 0$ and $a_1 + 2a_2 + 2a_3 < 1$. Iseki [9], Itoh [10].

(7) $H(fx, fy) \leqq a_1 d(x, y) + a_2 D(x, fx) + a_3 D(y, fy) + a_4 D(x, fy) + a_5 D(y, fx)$, $a_i \geqq 0$ for all i and $\min \{a_1 + a_2 + a_3 + 2a_4,\ a_1 + a_2 + a_3 + 2a_5\} < 1$. Kita [11].

(8) $H(fx, fy) \leqq a_1 \max \{d(x, y),\ D(x, fx),\ D(y, fy),\ \frac{1}{2}[D(x, fy) + D(y, fx)]\}$, $0 \leqq a_1 \leqq 1$. Ćirić [2].

Obviously, (1)-(7) respectively implies (8). We will show that (8) can be reformulated in our form. To begin with, let us see the following lemma;

Lemma. *Suppose that* (8) *holds for all x, y in X. Then for any $x \in X$, there exists a $y_x \in fx$ and*

$k>1$ *such that*

$$d(x, y_x) \leqq k(D(x, fx) - D(y_x, fy_x)).$$

Proof. Let $y \in fx$, then there exists $z \in fy$ such that

$$d(y, z) \leqq H(fx, fy) + \frac{1-a_1}{2} D(x, fx) \text{ (See Nadler [14])}$$

Since $y \in fx$ and $z \in fy$, $D(x, fx) \leqq d(x, y)$, $D(y, fy) \leqq d(y, z)$, $D(x, fy) \leqq d(x, z)$ and $D(y, fx) = 0$. Therefore by (8),

$$\begin{aligned} d(y, z) &\leqq H(fx, fy) + \frac{1-a_1}{2} D(x, fx) \\ &= a_1 \max\left\{ d(x, y),\ d(y, z),\ \frac{1}{2} d(x, z) \right\} + \frac{1-a_1}{2} D(x, fx) \\ &= a_1 \max\{d(x, y),\ d(y, z)\} + \frac{1-a_1}{2} D(x, fx) \end{aligned}$$

If $d(y, z) \geqq d(x, y)$, $d(y, z) \leqq a_1 d(y, z) + \frac{1-a_1}{2} d(y, z) = \frac{1+a_1}{2} d(y, z)$. But then $x = y = z$ is in fx, since $\frac{1+a_1}{2} < 1$. Suppose $d(x, y) \geqq d(y, z)$ then $d(y, z) \leqq a_1 d(x, y) + \frac{1-a_1}{2} d(x, y) = \frac{1+a_1}{2} d(x, y)$. In any case, $D(y, fy) \leqq d(y, z) \leqq \frac{1+a_1}{2} d(x, y)$. Since y was arbitrary, we can choose y_x in fx so that $d(x, y_x) \leqq \frac{a_1+3}{2a_1+2} D(x, fx)$ and $D(y_x, fy_x) \leqq \frac{1+a_1}{2} d(x, y_x)$. Let $k = \frac{2(a_1+3)}{(a_1+1)(1-a_1)} > 1$, then

$$\begin{aligned} &k(D(x, fx) - D(y_x, fy_x)) \\ &\geqq k\left(D(x, fx) - \frac{1+a_1}{2} d(x, y_x)\right) \\ &\geqq k\left(\frac{2a_1+2}{a_1+3} d(x, y_x) - \frac{1+a_1}{2} d(x, y_x)\right) \\ &= k \cdot \frac{1}{k} d(x, y_x) = d(x, y_x). \end{aligned}$$

This completes the proof.

From this lemma, we can set $\varphi(x) = kD(x, fx)$ and this φ and y_x satisfy (**). And the fact that φ is l.s.c. on $\overline{\{u_n\}}$ in our theorem is obvious.

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