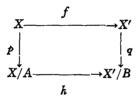
A Note on BCK-Algebras

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Abstract

(1) Let $f: X \to X'$ be a homomorphism of BCK-algebras and let A, B be ideals of X and X' respectively such that $f(A) \subset B$. Then there is a unique homomorphism $h: X/A \to X'/B$ such that the diagram



commutes.

(2) The class of all complexes of BCK-algebras becomes a category.

I. Introduction and Preliminaries

K. Iseki has introduced the notion of a *BCK*-algebra which is an algebraic formulation of a propositional calculus. In his various papers, Iseki has studied the structure of these algebras (see for example, [1], [2], [3] and [4]). In [6] D. Shin constructed the category on *BCK*-algebra and has studied the followings:

- (1) For each homomorphism $f: X \to X'$ of BCK-algebras with f(A) = 0, where A is an ideal in X, there is a unique homomorphism $\phi: X/A \to X'$ such that $f = \phi \circ \rho$.
 - (2) $p: X \rightarrow X/A$ is a universal element for a suitable functor.
- (3) If homorphism $f: X \rightarrow X'$ of BCK-algebras and A is an ideal of X', then $f^{-1}(A)$ is an ideal of X.

In this paper we find the induced homomorphism theorem on BCK-algebra and also we construct the new category on BCK-algebra.

Let us first recall a few fundamental definitions. Let X be a non-empty set with a binary operation * and suppose there is a constant 0 in X. Then (X, *, 0) (or simply denoted by X) is called a BCK-algebra if the following conditions hold:

$$(1) (x*y)*(x*z) \le z*y$$

$$(2) x*(x*y) \leq y$$

$$(3) x \leq x$$

$$(4) 0 \leq x$$

$$(5) x \le y \text{ and } y \le x \text{ imply } x = y$$

where $x \le y$ means x * y = 0.

For BCK-algebras X and X', a mapping $f: X \rightarrow X'$ is called a homomorphism if for any $x, y \in X$, f(x*y) = f(x)*f(y). If a homomorphism $f: X \rightarrow X'$ is onto, f is called an epimorphism. A non-empty subset A of a BCK-algebra X is called an ideal if the following conditions are satisfied:

- (1) 0∈*A*
- $(2) x \in A \text{ and } y*x \in A \text{ imply } y \in A.$

We shall state some properties on BCK-algebra:

- $(1) x \le y \text{ implies } z * y \le z * x$
- $(2) x \leq y, y \leq z \text{ implies } x \leq z$
- $(3) (x*y)*z \leq (z*x)*y$
- (4) (x*y)*z=(x*z)*y
- $(5) (x*y) \le z \text{ implies } x*z \le y$
- $(6) (x*y)*(z*y) \leq x*z$
- $(7) x \le y \text{ implies } x * z \le y * z$
- $(8) x*y \le x$
- (9) x*0=x for all x, y, z in BCK-algebra X.

Theorem 1. ([2]) Let $f: X \rightarrow X'$ be a homomorphism. Then the kernel of f, Ker(f), is an ideal of X.

Theorem 2. ([2]) If A is an ideal of BCK-algebra X, then the quotient X/A is also a BCK-algebra.

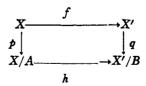
Theorem 3. Let $f: X \rightarrow X'$ be a homomorphism. Then $p: X \rightarrow X/Ker(f)$ is a homomorphism.

Theorem 4. ((3)) Let X, Y, Z be BCK-algebras, and let $h: X \rightarrow Y$ be an epimorphism, and let $g: X \rightarrow Z$ be a homomorphism. If $Ker(h) \subset Ker(g)$, then there is a unique homomorphism $f: Y \rightarrow Z$ satisfying $f \circ h = g$.

II. Results

We shall prove Induced homomorphism theorem on BCK-algebra.

Theorem 1. (Induced homomorphism theorem) Let $f: X \rightarrow X'$ be a homomorphism of BCK-algebras and let A and B be ideals of X and X' respectively such that $f(A) \subset B$. Then there is a unique homomorphism $h: X/A \rightarrow X'/B$ such that the diagram



commutes.

Proof. Define $h: X/A \to X'/B$ by h(Ax) = Bf(x) where $x \in X$. To prove that h is well defined, let $Ax = Ay(x, y \in X)$, then $x*y \in A$. Then $f(x*y) \in f(A)$. Since f is homomorphism and since $f(A) \subset B$, $f(x)*f(y) \in B$. Hence Bf(x) = Bf(y) i.e. h(Ax) = h(Ay). To prove h is a homomorphism, let $Ax, Ay \in X/A$. Then h(Ax*Ay) = h(A(x*y)) = Bf(x*y) = B(f(x)*f(y)) = Bf(x)*Bf(y) = h(Ax)*h(Ay). To prove the commutativity of the diagram, let $Ax \in X/A$. Then $(h \circ p)(x) = h(p(x)) = h(Ax) = Bf(x) = q(f(x)) = (q \circ f)(x)$ i.e. $h \circ p = q \circ f$. To verify the uniqueness of h, let $h \in X/A \to X'/B$ be a homomorphism such that $h \circ p = q \circ f$. For $Ax \in X/A(x \in X)$, $h(Ax) = h(p(x)) = (h \circ p)(x) = (q \circ f)(x) = (h \circ p)(x) = h(Ax)$ and hence $h \in X/A$.

Definition. By a complex of BCK-algebras, we mean a family $X = \{X_i, d_i\}_{i \in I}$ of BCK-algebras X_i and homomorphisms $d_i : X_i \to X_{i-1}$ of BCK-algebras such that $d_{i-1}d_i = 0$ for all i.

Let $X = \{X_i, d_i\}$, $X' = \{X_i', d_i'\}$ be complexes of BCK-algebras. We define a morphism $f: X \to X'$ to be a family $\{f_i: X_i \to X_i'\}$ of homorphisms of BCK-algebras such that $f_i d_{i+1} = d_{i+1} f_{i+1}$ for all i. Then for $f \in hom(X, X_i)$, $g \in hom(X', X'')$, $gf = \{g_i f_i: X_i \to X_i''\} \in hom(X, X'')$ and obviously associativity holds. Define $1_X: X \to X$ by $\{1_{X_i}: X_i \to X_i\}$. Then $1_X f = f = f 1_X$. Hence we have the following

Theorem 2. The class of all complexes of BCK-algebras becomes a category.

References

- 1. K. Iseki, Some Properties of BCK-algebras, Math. Seminar Notes 2(1974), Kobe Univ., 193-201.
- 2. K. Iseki, On Ideals in BCK-algebras, Math. Seminar Notes 3(1975). Kobe Univ., 1-12.
- 3. K. Iseki, Remarks on BCK-algebras, Math. Seminar Notes 3 (1975). Kobe Univ., 45-53.
- 4. K. Iseki, On some Ideals in BCK-algebras, Math. Seminar Notes 3(1975), Kobe Univ., 65-70.
- 5. K. Iseki, BCK-algebra, Math. Seminar Notes 4(1976), Kobe Univ., 77-86.
- 6. D.S. Shin, On the BCK-algebra, J. of Kor. Res. Inst. Liv'., 19(1977), Ewha Womans Univ., 11-16.