韓國數學教育學會誌<數學教育>
1983. 12. Vol. XXII, No. 1.

Journal of the Korea Society of Mathematical Education
Dec. 1983. Vol. XXII, No. 1.

A Note on BCK-Algebras

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Abstract

(1) Let $f: X \to X'$ be a homomorphism of BCK-algebras and let A, B be ideals of X and X' respectively such that $f(A) \subset B$. Then there is a unique homomorphism $h: X/A \to X'/B$ such that the diagram

$$\begin{array}{ccc} X & \xrightarrow{\quad f \quad} & X' \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q} \\ X/A & \xrightarrow[\quad h \quad]{} & X'/B \end{array}$$

commutes.

(2) The class of all complexes of BCK-algebras becomes a category.

I. Introduction and Preliminaries

K. Iseki has introduced the notion of a BCK-algebra which is an algebraic formulation of a propositional calculus. In his various papers, Iseki has studied the structure of these algebras (see for example, [1], [2], [3] and [4]). In [6] D. Shin constructed the category on BCK-algebra and has studied the followings:

(1) For each homomorphism $f: X \to X'$ of BCK-algebras with $f(A)=0$, where A is an ideal in X, there is a unique homomorphism $\phi: X/A \to X'$ such that $f=\phi \circ p$.

(2) $p: X \to X/A$ is a universal element for a suitable functor.

(3) If homorphism $f: X \to X'$ of BCK-algebras and A is an ideal of X', then $f^{-1}(A)$ is an ideal of X.

In this paper we find the induced homomorphism theorem on BCK-algebra and also we construct the new category on BCK-algebra.

Let us first recall a few fundamental definitions. Let X be a non-empty set with a binary operation $*$ and suppose there is a constant 0 in X. Then $(X, *, 0)$ (or simply denoted by X) is called a BCK-algebra if the following conditions hold:

(1) $$(x*y)*(x*z) \leq z*y$$

(2) $$x*(x*y) \leq y$$

(3) $x \leq x$

(4) $0 \leq x$

(5) $x \leq y$ and $y \leq x$ imply $x = y$

where $x \leq y$ means $x * y = 0$.

For BCK-algebras X and X', a mapping $f: X \rightarrow X'$ is called a homomorphism if for any $x, y \in X$, $f(x*y) = f(x) * f(y)$. If a homomorphism $f: X \rightarrow X'$ is onto, f is called an epimorphism. A non-empty subset A of a BCK-algebra X is called an ideal if the following conditions are satisfied:

(1) $0 \in A$

(2) $x \in A$ and $y * x \in A$ imply $y \in A$.

We shall state some properties on BCK-algebra:

(1) $x \leq y$ implies $z*y \leq z*x$

(2) $x \leq y$, $y \leq z$ implies $x \leq z$

(3) $(x*y)*z \leq (z*x)*y$

(4) $(x*y)*z = (x*z)*y$

(5) $(x*y) \leq z$ implies $x*z \leq y$

(6) $(x*y)*(z*y) \leq x*z$

(7) $x \leq y$ implies $x*z \leq y*z$

(8) $x*y \leq x$

(9) $x*0 = x$ for all x, y, z in BCK-algebra X.

Theorem 1. ([2]) *Let $f: X \rightarrow X'$ be a homomorphism. Then the kernel of f, $Ker(f)$, is an ideal of X.*

Theorem 2. ([2]) *If A is an ideal of BCK-algebra X, then the quotient X/A is also a BCK-algebra.*

Theorem 3. *Let $f: X \rightarrow X'$ be a homomorphism. Then $p: X \rightarrow X/Ker(f)$ is a homomorphism.*

Theorem 4. ([3]) *Let X, Y, Z be BCK-algebras, and let $h: X \rightarrow Y$ be an epimorphism, and let $g: X \rightarrow Z$ be a homomorphism. If $Ker(h) \subset Ker(g)$, then there is a unique homomorphism $f: Y \rightarrow Z$ satisfying $f \circ h = g$.*

II. Results

We shall prove Induced homomorphism theorem on BCK-algebra.

Theorem 1. (Induced homomorphism theorem) *Let $f: X \rightarrow X'$ be a homomorphism of BCK-algebras and let A and B be ideals of X and X' respectively such that $f(A) \subset B$. Then there is a unique homomorphism $h: X/A \rightarrow X'/B$ such that the diagram*

$$\begin{array}{ccc} X & \xrightarrow{\quad f \quad} & X' \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q} \\ X/A & \xrightarrow[\quad h \quad]{} & X'/B \end{array}$$

commutes.

Proof. Define $h: X/A \to X'/B$ by $h(Ax) = Bf(x)$ where $x \in X$. To prove that h is well defined, let $Ax = Ay (x, y \in X)$, then $x*y \in A$. Then $f(x*y) \in f(A)$. Since f is homomorphism and since $f(A) \subset B$, $f(x)*f(y) \in B$. Hence $Bf(x) = Bf(y)$ i.e. $h(Ax) = h(Ay)$. To prove h is a homomorphism, let $Ax, Ay \in X/A$. Then $h(Ax*Ay) = h(A(x*y)) = Bf(x*y) = B(f(x)*f(y)) = Bf(x)*Bf(y) = h(Ax)*h(Ay)$. To prove the commutativity of the diagram, let $Ax \in X/A$. Then $(h \circ p)(x) = h(p(x)) = h(Ax) = Bf(x) = q(f(x)) = (q \circ f)(x)$ i.e. $h \circ p = q \circ f$. To verify the uniqueness of h, let $k: X/A \to X'/B$ be a homomorphism such that $k \circ p = q \circ f$. For $Ax \in X/A (x \in X)$, $k(Ax) = k(p(x)) = (k \circ p)(x) = (q \circ f)(x) = (h \circ p)(x) = h(p(x)) = h(Ax)$ and hence $k = h$.

Definition. By a *complex* of *BCK*-algebras, we mean a family $X = \{X_i, d_i\}_{i \in I}$ of *BCK*-algebras X_i and homomorphisms $d_i: X_i \to X_{i-1}$ of *BCK*-algebras such that $d_{i-1}d_i = 0$ for all i.

Let $X = \{X_i, d_i\}$, $X' = \{X_i', d_i'\}$ be complexes of *BCK*-algebras. We define a morphism $f: X \to X'$ to be a family $\{f_i: X_i \to X_i'\}$ of homorphisms of *BCK*-algebras such that $f_i d_{i+1} = d_{i+1} f_{i+1}$ for all i. Then for $f \in hom(X, X_i)$, $g \in hom(X', X'')$, $gf = \{g_i f_i: X_i \to X_i''\} \in hom(X, X'')$ and obviously associativity holds. Define $1_X: X \to X$ by $\{1_{X_i}: X_i \to X_i\}$. Then $1_X f = f = f 1_X$. Hence we have the following

Theorem 2. *The class of all complexes of BCK-algebras becomes a category.*

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