⟨Research Paper⟩

Journal of the Korean
Statistical Society
Vol. 12, No. 2, 1983

Central Limit Theorem for Lèvy Processes

In Suk Wee*

ABSTRACT

Let $\{X_t\}$ be a process with stationary and independent increments whose log characteristic function is expressed as $ibut-2^{-1}\sigma^2u^2t+t\int_{\{0\}^c}(\exp(iux)-1-iux(1+x^2)^{-1})d\nu(x)$. Our main result is that $x^2\left(\int_{|y|>x}d\nu(y)\right)/\left(\int_{|y|\le x}y^2d\nu(y)+\sigma^2\right)\to 0$ as $x\to 0$ (resp. $x\to\infty$) is necessary, and sufficient for $\{X_t\}$ to have $\{A_t\}$ and $\{B_t\}$ such that $(X_t-A_t)/\ B_t\xrightarrow{\mathscr{D}}n(0,1)$ as $t\to 0$ (resp. $t\to\infty$).

1. Introduction

Let $\{X_t\}$ be a process with stationary, independent increments taking values in R^1, whose log characteristic function is expressed as

$$(1.1)\qquad ibut-2^{-1}\sigma^2u^2t+t\int_{\{0\}^c}(\exp(iux)-1-iux(1+x^2)^{-1})d\nu(x)$$

where ν is the Lèvy measure satisfying

$$\int x^2/(1+x^2)d\nu(x)<\infty.$$

The problem we consider in this paper is to find a necessary and sufficient condition for Lèvy process $\{X_t\}$ to have $\{A_t\}$ and $\{B_t\}$ such that as $t\to 0$ and ∞,

$$(X_t-A_t)B_t^{-1}\xrightarrow{\mathscr{D}}n(0,1),$$

where $\xrightarrow{\mathscr{D}}$ denotes convergence in distribution and $n(0,1)$ represents a standard normal random variable.

It is known that for a distribution function $F(x)$ to belong to the domain of attraction of normal law, it is necessary and sufficient that as $x\to\infty$,

$$x^2\left(\int_{|y|>x}dF(y)\right)/\left(\int_{|y|\le x}y^2dF(y)\right)\to 0.$$

*Korea University

(See Gnedenko and Kolmogorov (1968), p. 172.)

We are able to get the analogue for this result replacing the distribution function with Lèvy measure both for $t\to 0$ and $t\to\infty$. In fact, our result is that, as $x\to 0$ and ∞,

$$x^2\left(\int_{|y|>x} d\nu(y)\right)\Big/\left(\int_{|y|\le x} y^2 d\nu(y)+\sigma^2\right)\to 0 \tag{1.2}$$

iff there are $\{A_t\}$ and $\{B_t\}$ such that

$$(X_t-A_t)/B_t \xrightarrow{\mathscr{D}} \mathrm{n}(0,1)$$

as t tends to 0 and ∞ respectively.

There has been no attempt to characterize the Central Limit Theorem by Lèvy measure and this work is motivated by Pruitt (1981) which shows the similar role played by distribution function for random walk and by Lèvy measure for Lèvy process. The technique used here is similar to the one in Kolmogorov and Gnedenko (1968).

But the main difficulty in our problem is that ν is not a finite measure in general, and we are considering the case both for small and large t. It turns out that both for $t\to 0$ and $t\to\infty$, $B_t\sim t^{1/2}L(t)$ where $L(t)$is slowly varying at 0 and ∞. The centering is irrelevant for small t, since we can easily obtain that as $t\to 0$

$$A_t/B_t\to 0.$$

But the situation for $t\to\infty$ is somewhat different. In fact, if (1.2) holds as $x\to\infty$, then $EX_1<\infty$ and $A_t\sim tEX_1$, since by Lemma 2.2(2), $\int x^3/(1+x^2)d\nu(x)<\infty$ and by differentiating the characteristic function, $EX_t=bt+t\int x^3/(1+x^2)d\nu(x)$. Thus unless the mean of X_1 is zero, we have as $t\to\infty$, $A_t/B_t\to\pm\infty$.

2. Preliminaries

We define, for $x>0$,

$$G(x)=\int_{|y|>x} d\nu(y),$$

$$K(x)=x^{-2}\left(\int_{|y|\le x} y^2 d\nu(y)+\sigma^2\right),$$

$$M(x)=\int_{|y|\le x} y^3(1+y^2)^{-1} d\nu(y),$$

$$N(x)=\int_{|y|>x} y(1+y^2)^{-1} d\nu(y).$$

We prove two lemmas which will be used for main results.

Lemma 2.1.

(1) If $\lim_{x\to 0} G(x)/K(x)=0$, then $x^2K(x)$ is slowly varying at 0.

(2) If $\lim_{x\to\infty} G(x)/K(x)=0$, then $x^2K(x)$ is slowly varying at ∞.

Proof. For $\eta>1$, we have

$$\frac{\int_{x<|y|\leq\eta x} y^2 d\nu(y)}{\int_{|y|\leq x} y^2 d\nu(y)+\sigma^2}\leq\frac{\eta^2 G(x)}{K(x)}$$

which implies the slow variation of $x^2K(x)$ both at 0 and ∞.

Lemma 2.2.

(1) If $\lim_{x\to 0}\frac{G(x)}{K(x)}=0$, then $\lim_{x\to 0}\frac{N(x)}{xK(x)}=0$.

(2) If $\lim_{x\to\infty}\frac{G(x)}{K(x)}=0$, then

$$\overline{M}(x)=\int_{|y|>x} y^3/(1+y^2)d\nu(y)$$

converges, and

$$\lim_{x\to\infty}\frac{\overline{M}(x)}{xK(x)}=0.$$

Proof.

(1) By integrating by parts, we have for $0<x<1$,

(2.1) $$|N(x)-N(1)|\leq\int_{x<|y|\leq 1}|y|d\nu(y)$$
$$=K(1)-xK(x)+\int_{x<y\leq 1}K(y)dy.$$

Define,

$$l(y)=\begin{cases} y^2K(y) & \text{if} \quad 0<y\leq 1 \\ 0 & \text{if} \quad y>1. \end{cases}$$

Then l is slowly varying at 0. Making the change of variable, $y=1/z$, we have

$$l(y)=l(1/z)=L(z)$$

where L is slowly varying at ∞. Also

(2.2) $$\int_{x<y\leq 1}K(y)dy=\int_{x<y} y^{-2}l(y)dy=\int_{x^{-1}<z}L(z)dz$$

and for $x<1$,

(2.3) $$xK(x)=x^{-1}L(x^{-1}).$$

By using theorem 1 of Feller (1966, p. 281), (2.2) and (2.3), we have as $x\to 0$,

$$\frac{\int_{x<y\leq 1}K(y)dy}{xK(x)}=\frac{\int_{x^{-1}<z}L(z)dz}{x^{-1}L(x^{-1})}\to 1.$$

Combining this with (2.1), we have as $x\to 0$,

$$\frac{N(x)}{xK(x)}\to 0,$$

since $xK(x)=l(x)/x\to\infty$.

(2) This can be proved similarly.

3. Main Results

Our final result will be obtained by proving two theorems, Theorem 3.1 and 3.2.

Theorem 3.1.

(1) $\lim_{x\to 0}G(x)/K(x)=0$ iff there exists $\{B_t\}$ such that for any $\varepsilon>0$, as $t\to 0$,

(3.1) $$B_t\to 0,$$

(3.2) $$\varepsilon^2 tK(\varepsilon B_t)\to 1,$$

(3.3) $$tG(\varepsilon B_t)\to 0.$$

In this case, for any $\varepsilon>0$, as $t\to 0$,

(3.4) $$tN(\varepsilon B_t)/B_t\to 0.$$

(2) $\lim_{x\to\infty}G(x)/K(x)=0$. iff there exists $\{B_t\}$ such that for any $\varepsilon>0$, as $t\to\infty$,

(3.5) $$B_t\to\infty,$$

(3.6) $$\varepsilon^2 tK(\varepsilon B_t)\to 1,$$

(3.7) $$tG(\varepsilon B_t)\to 0.$$

In this case, for any $\varepsilon>0$, as $t\to\infty$,

(3.8) $$t\overline{M}(\varepsilon B_t)/B_t\to 0.$$

Proof. (1) Let $B_t=\inf\{x: K(x)\leq 1/t\}$ for t sufficiently small. Then we have $B_t\to 0$ as $t\to 0$. Because of the slow variation of $x^2K(x)$ at 0, we have $B_t>0$. Since $x^2K(x)$ is nondecreasing, for $0<\delta<1$, we have

$$tK(B_t)\geq(1-\delta)^2 tK((1-\delta)B_t)>(1-\delta^2).$$

From this and the right continuity of $K(x)$, we obtain, for t sufficiently small,

$$tK(B_t)=1.$$

Furthermore, since $x^2K(x)$ is slowly varying at 0, we have for any $\varepsilon>0$, as $t\to 0$,

$$(\varepsilon B_t)^2K(\varepsilon B_t)/(B_t^2K(B_t))\to 1,$$

so we get (3.2). For the proof of (3.3), we use

$$tG(\varepsilon B_t)=\frac{G(\varepsilon B_t)}{K(\varepsilon B_t)}tK(\varepsilon B_t)$$

which tends to 0 as $t\to 0$. Conversely, suppose that $\{B_t\}$ satisfying (3.1), (3.2) and (3.3) exists. Let $B_{1/(n+1)}<x\leq B_{1/n}$, and $B^{n+1}=B_{1/(n+1)}$, $B^n=B_{1/n}$.

Then we get

$$G(B^{n+1})\geq G(x)\geq G(B^n)$$
$$K(B^{n+1})+G(B^{n+1})\geq K(x)\geq K(B^n)-G(B^{n+1}),$$

so

$$\frac{(1/n)G(B^n)}{((n+1)/n)[(1/(n+1))K(B^{n+1})+(1/(n+1))G(B^{n+1})]}\leq\frac{G(x)}{K(x)}$$
$$\leq\frac{(1/(n+1))G(B^{n+1})}{(n/(n+1))(1/n)K(B^n)-(1/(n+1))G(B^{n+1})}$$

which proves the assertion.

To prove (3.4), setting $x=\varepsilon B_t$ in Lemma 2.2(1), we have, as $t\to 0$,

$$\frac{tN(\varepsilon B_t)}{B_t}=\frac{N(\varepsilon B_t)}{\varepsilon B_t K(\varepsilon B_t)}\varepsilon tK(\varepsilon B_t)\to 0.$$

(2) This can be proved in the same way.

Theorem 3.2.

(1) $\lim_{x\to 0}G(x)/K(x)=0$ iff there exists $\{B_t\}$ such that as $t\to 0$,

$$B_t^{-1}X_t\xrightarrow{\mathscr{D}}n(0,1).$$

(2) $\lim_{x\to\infty}G(x)/K(x)=0$ iff there exist $\{A_t\}$ and $\{B_t\}$ such that as $t\to\infty$,

$$B_t^{-1}(X_t-A_t)\xrightarrow{\mathscr{D}}n(0,1).$$

Proof. (1) We will prove that there exists $\{B_t\}$ for which (3.1), (3.2) and (3.3) hold iff $B_t^{-1}X_t\to n(0,1)$ as $t\to 0$. Then by Theorem 3.1, we are finished. Suppose that there exists such $\{B_t\}$. Let $A_t=(M(B_t)+b)t$. We use the log characteristic function of $B_t^{-1}(X_t-A_t)$;

(3.9) $$-\frac{\sigma^2u^2}{2B_t^2}t+t\int_{|x|\leq B^t}(\exp(iuxB_t^{-1})-1-iuxB_t^{-1})d\nu(x)$$
$$+t\int_{|x|>B_t}(\exp(iuxB_t^{-1})-1)d\nu(x)-iuxN(B_t)B_t^{-1}.$$

Let u be fixed. The first two terms in (3.9) can be written as

(3.10) $$-\frac{\sigma^2u^2}{2B_t^2}t+t\int_{|x|\leq B_t}\Big(-\frac{u^2x^2}{2B_t^2}+\theta((ux/B_t)^2)\Big)d\nu(x)$$

where $\theta(y)/y\to 0$ as $y\to 0$ and θ is bounded on compact sets. By (3.2), as $t\to 0$,

$$-\frac{\sigma^2u^2}{2B_t^2}t+t\int_{|x|\leq B^t}-\frac{u^2x^2}{2B_t^2}d\nu(x)=-u^2tK(B_t)/2\to-\frac{u^2}{2}.$$

Because of (3.2) and (3.3), for any given $\eta>0$, it is possible to choose $\varepsilon<1$, $t(\varepsilon,\eta)$ such that for $|y|<\varepsilon$,

$$\theta(u^2y^2)\leq\eta u^2y^2$$

and for $t<t(\varepsilon,\eta)$

$$tG(\varepsilon B_t)\leq\eta,\quad \varepsilon^2tK(\varepsilon B_t)\leq 2.$$

From these, we have for $t<t(\varepsilon,\eta)$,

$$\begin{aligned}t\int_{|x|\leq B_t}\theta((ux/B_t)^2)d\nu(x)&=t\int_{|x|\leq\varepsilon B_t}\theta((ux/B_t)^2)d\nu(x)\\&\quad+t\int_{\varepsilon B_t<|x|\leq B_t}\theta((ux/B_t)^2)d\nu(x)\\&\leq 2\eta u^2+c\eta.\end{aligned}$$

Thus we have proved that for any u, (3.10) converges to $-u^2/2$ as $t\to 0$. By Lemma 2.2(1), we have as $t\to 0$,

$$tuN(B_t)/B_t\to 0.$$

so we complete the proof of the convergence of $B_t^{-1}(X_t-A_t)$ to $n(0,1)$ by showing that the second integral in (3.9) converges to 0 as $t\to 0$. By (3.3), we have

$$t\left|\int_{|x|>\varepsilon B_t}(\exp(iux/B_t)-1)d\nu(x)\right|\leq 2tG(\varepsilon B_t)\to 0,$$

as $t\to 0$. Next for the proof of $B_t^{-1}A_t\to 0$, (3.2) and the slow variation of $x^2K(x)$ at 0 imply that as $t\to 0$,

$$t/B_t\sim B_t/(B_t^2K(B_t))\to 0$$

Thus we have, as $t\to 0$,

$$\begin{aligned}|B_t^{-1}A_t|&=t(M(B_t)+b)/B_t\\&\leq tB_t^2K(B_t)+t|b|/B_t\to 0.\end{aligned}$$

Conversely, suppose that

$$B_t^{-1}X_t\xrightarrow{\mathscr{D}}n(0,1).$$

so for any u, as $t\to 0$,

$$-\frac{\sigma^2u^2}{2B_t^2}t+i\frac{but}{B_t}+t\int\left(\exp(iux/B_t)-1-\frac{iux}{B_t(1+x^2)}\right)d\nu(x)\to-\frac{u^2}{2}.$$

Therefore as $t\to 0$,

$$-\frac{\sigma^2u^2}{2B_t^2}t+t\int(\cos(ux/B_t)-1)d\nu(x)\to-\frac{u^2}{2}.$$

Define, for any Borel set A for which $0\notin A$,

$$\nu_t(A)=t\nu(B_tA).$$

$$\gamma_t(A)=\int_A x^2/(1+x^2)d\nu_t(x)$$

and

$$\gamma_t\{0\}=t\sigma^2/B_t^2.$$

Then

$$-\frac{\sigma^2u^2}{2B_t^2}t+t\int(\cos(ux/B_t)-1)d\nu(x)$$

$$=\int(\cos(ux)-1)(1+x^2)/x^2d\gamma_t(x)$$

where

$$\Big[(\cos(ux)-1)(1+x^2)/x^2\Big]_{x=0}=-u^2/2.$$

We will prove that for $\{t_k\}$, $t_k\uparrow\infty$, $\{r_{t_k}\}$ are conditionally compact under weak convergence. For k sufficiently large,

$$\int(1-\cos ux)(1+x^2)/x^2dr_{t_k}(x)\leq u^2/2+\varepsilon.$$

Thus we have, for k sufficiently large,

$$3^{-1}\int_{|x|\leq 1}dr_{t_k}(x)\leq 3^{-1}\int_{|x|\leq 1}(1+x^2)dr_{t_k}(x)\leq 1/2+\varepsilon.$$

Also

$$\int_{|x|>1}\left(\int_0^2(1-\cos ux)du\right)(1+x^2)/x^2dr_{t_k}(x)\leq\int_0^2(u^2/2+\varepsilon)du,$$

so

$$\frac{1}{2}\int_{|x|>1}dr_{t_k}(x)\leq\int_{|x|>1}\left(1-\frac{\sin 2x}{2x}\right)(1+x^2)/x^2dr_{t_k}(x)\leq 4/3+2\varepsilon.$$

We have proved that $\{r_{t_k}\}$ are uniformly bounded. There exists $\alpha>0$ such that for any t,

$$\int\left(\int_0^{1/T}(1-\cos ux)du\right)(1+x^2)/x^2dr_t(x)\geq\frac{\alpha}{T}\int_{|x|\geq T}dr_t(x).$$

Therefore, by Fatou's lemma,

$$\overline{\lim_{k\to\infty}}\int_{|x|\geq T}dr_{t_k}(x)\leq\overline{\lim_{k\to\infty}}\frac{T}{\alpha}\int_0^{1/T}\left(\int(1-\cos ux)(1+x^2)/x^2dr_{t_k}(x)\right)du$$

$$\leq\frac{T}{\alpha}\int_0^{1/T}u^2/2du$$

$$=(6\alpha T^2)^{-1}.$$

This proves that $\{r_{t_k}\}$ are conditionally compact. Suppose that for some $\{t_k'\}\subset\{t_k\}$,

$$\gamma_{t_k'} \overset{w}{\Longrightarrow} r^*.$$

Then along $\{t_k'\}$,

$$(3.11)\qquad -\frac{\sigma^2 u^2}{2B_t^2}t + t\int\left(\exp(iux/B_t) - 1 - \frac{iux/B_t}{1+(x/B_t)^2}\right)d\nu(x)$$

$$= \int(\exp(iux) - 1 - iux(1+x^2)^{-1})(1+x^2)/x^2 dr_t(x)$$

$$\rightarrow \int(\exp(iux) - 1 - iux(1+x^2)^{-1})(1+x^2)/x^2 dr^*(x).$$

On the other hand, along $\{t_k'\}$,

$$(3.12)\qquad -\frac{\sigma^2 u^2}{2B^2_t}t + \frac{iu(bt - A_t)}{B_t} + t\int\exp(iux/B_t) - 1 - iux(1+x^2)^{-1}B_t^{-1})d\nu(x)$$

$$\rightarrow -u^2/2.$$

By (3.11), (3.12) and the uniqueness of Lèvy representation,

$$\int(\exp(iux) - 1 - iux(1+x^2)^{-1})(1+x^2)x^{-2}dr^*(x) = -u^2/2,$$

so $r^*\{0\} = 1$, and $r^*((-\infty, 0)\cup(0, \infty)) = 0$.

Since this is possible for any subsequence $\{t_k'\}$, we have

$$r_{t_k} \overset{w}{\Longrightarrow} r^*.$$

Thus for any $\varepsilon > 0$,

$$\varepsilon^2 t_k K(\varepsilon B_{t_k}) = t_k\left(\sigma^2 + \int_{|x|\leq \varepsilon B t_k} x^2 d\nu(x)\right)/B_{t_k}^2$$

$$= \int_{|x|\leq\varepsilon}(1+x^2)dr_{t_k}(x) \rightarrow r^*\{0\} = 1,$$

and

$$t_k G(\varepsilon B_{t_k}) = t_k\int_{|x|>\varepsilon B t_k} d\nu(x)$$

$$= \int_{|x|>\varepsilon} d\nu_{t_k}(x)$$

$$= \int_{|x|>\varepsilon}(1+x^2)/x^2 d\gamma_{t_k}(x)$$

$$\rightarrow \int_{|x|>\varepsilon}(1+x^2)/x^2 dr^*(x) = 0.$$

Therefore, we have, for any $\varepsilon > 0$, as $t \rightarrow 0$.

$$\varepsilon^2 t K(\varepsilon B_t) \rightarrow 1$$

and

$$tG(\varepsilon B_t) \rightarrow 0.$$

(2) This can be proved basically in the same way, but only difference is that we

are not able to get, as $t\to\infty$, $B_t^{-1}A_t\to 0$. Instead, we can obtain that EX_1 exists and if $EX_1=0$, then as $t\to\infty$, $A_t/B_t\to 0$, using Lemma 2.2(2).

Remark 1.

It remains open whether or not the analogue of our result holds for the convergence to a stable distribution. There is an analogous condition for a distribution to belong to the domain of attraction of stable law, so our conjecture is that there would be some similar conditions under which

$$B_t^{-1}(X_t-A_t)\xrightarrow{\mathscr{D}} Y$$

where Y has a stable distribution.

Remark 2.

It is easy to check that for Brownian Motion $\{X_t\}$, (1.2) holds for $x\to 0$ and ∞, and $A_t=bt$, $B_t=\sqrt{t}\,\sigma$. For Compound Poisson Process with log characteristic function $\lambda t(\varphi(u)-1)$ where φ is Characteristic function of a distribution function F,

$$\nu(B)=\lambda F(B\cap\{0\}^c) \text{ and } \sigma^2=0,$$

so

$$\frac{G(x)}{K(x)}=\frac{x^2\int_{|y|>x}dF(y)}{\int_{|y|\le x}y^2dF(y)}$$

In particular, for Poisson Process, (1.2) holds for $x\to\infty$, but not for $x\to 0$. Thus for $t\to\infty$,

$$B_t=\sqrt{\lambda t},\ \ A_t=\lambda t.$$

For another exxample, consider Lèvy Process with Lèvy measure given by

$$d\nu(x)=\frac{1}{|x|^3(\log^2|x|+1)}dx$$

and $\sigma^2=0$.

Then for x large and small,

$$G(x)\sim\frac{1}{x^2(\log^2 x)}$$

and

$$K(x)\sim\begin{cases} C/x^2 & \text{for } x \text{ large} \\ 1/(x^2|\log x|) & \text{for } x \text{ small}\end{cases}$$

where C is a finite and positive constant. So (1.2) holds both for $x\to 0$ and ∞ and for $t\to 0$, $B_t=\sqrt{t}L(t)$ where $L(t)$ is a slowly varying function at 0, and for $t\to\infty$, $B_t\sim\sqrt{Ct}$, and $A_t\sim bt$.

REFERENCES

(1) Feller (1966). *An Introduction to Probability Theory and Its Application*, Vol. II, Wiley, New York.

(2) Gnedenko and Kolmogorov (1968). *Limit distributions for Sums of Independent Random Variables*, Addison-Wesley Publishing Company Inc.

(3) Pruitt (1981). The Growth of Random Walks and Lèvy Processes, *Annals of Probability*, Vol. 9, No. 6, 948—956.