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Estimation of the Scale Parameter in the Weibull Distribution Based on the Quasi-range

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ABSTRACT

The purpose of this paper is to obtain representation of the mathematical special functions and the numerical values of the mean square errors for the quasi-ranges in random small samples($n \leq 30$) from the Weibull distribution with a shape and a scale parameters, and to estimate the scale parameter by use of unbiased estimator based on the quasi-range. It will be shown that the jackknife estimator of the range is worse than the range of random samples from the given distribution in the sense of the mean square error.

1. Introduction

Consider the Weibull distribution with two parameters defined by the distribution function

$$F_x(x)=\begin{cases}1-\exp\{-(x/\beta)^{\alpha}\}, & x>0\\ 0, & \text{elsewhere,}\end{cases} \tag{1.1}$$

where α is a shape parameter, β is a scale parameter,

$$\alpha>0 \text{ and } \beta>0.$$

It is well known that this distribution has mean $\beta\Gamma(1+1/\alpha)$ and variance $\beta^2\{\Gamma(1+2/\alpha)-\Gamma^2(1+1/\alpha)\}$. Some remarks on history and a good overview of the applications of the distribution may be found in Mann(1968) and Mann et al. (1974), respectively. Order statistic from the Weibull distribution with only a shape parameter has been considered by Lieblein(1955). He obtained the first two moments of the order statistic in terms of incomplete beta and gamma functions in itself.

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The r-th quasi-range, say W_r, of a sample of size n is defined as the range of $(n-2r)$ sample values deleting the r largest and the r smallest samples. That is,

$$W_r = X_{(n-r)} - X_{(r+1)}, \quad (r < (n-1)/2);$$

where $X_{(1)}$, $X_{(2)}, , X_{(n)}$ are the order statistics of a random sample of size n from the given distribution.

A number of authors, including Rider (1959), Govindarajulu (1963), Singh(1970), Malik(1980), Gupt and Singh (1981), Lee and Kapadia(1982), have been concerned with the properties of the quasi-ranges from the normal, the nonnormal, the exponential, the rectangle, or the logistic distribution.

In section 2, the first two moments of the order statistics from the distribution are represented as the mathematical special functions by use of a suitable integral technique(c.f. [4]). In section 3, we shall obtain the mean, variance and mean square error of the rth quasi-range of random samples from the distribution by the results in section 2, and evaluate the exact numerical values of the mean square errors for $r=0(1)$ $[(n-1)/2]$ and $n=5(5)30$ by use of computer. In section 4, we shall obtain a uniformly minimum variance unbiased estimator of the scale parameter when the shape parameter is known, and consider the efficiency (percent) of an unbiased estimator of the scale parameter based on the r-th quasi-range of random samples from the distribution, and evaluate the exact numerical values of the efficiencies for $r=0(1)$ $[(n-1)/2]$ and $n=5(5)30$ by use of computer. Finally, in section 5 we shall compare the exact numerical values of the mean square error of the range with those of the mean square error of the jackknife estimator for the range from the distribution for $\alpha=1/3$, $1/2$, 1, 2 and $n=5(5)30$.

The relative mean square error occured in the tables is defined as the mean square error divided by the unknown parameter.

2. Moment of the Order Statistic

Let $f_x(x)$ be a probability density function with the distribution (1.1). It is well known that the probability density function of the p-th order statistic and the joint probability density function of the p-th and q-th order statistics are represented by the population distribution function and its probability density function as follow;

$$f_{x(p)}(x) = \frac{n!}{(p-1)!(n-p)!}[F_x(x)]^{p-1}[1-F_x(x)]^{n-p}f_x(x); \; 0<x<\infty.$$

and

$$f_{x_{(p)}x_{(q)}}(x,y)=\frac{n!}{(p-1)!(q-p-1)!(n-q)!}[F_x(x)]^{p-1}$$
$$[F_x(y)-F_x(x)]^{q-p-1}[1-F_x(y)]^{n-q}f_x(x)f_x(y),$$
$$0<x\leq y<\infty.$$

For convenience, let us denote $n!/\{(p-1)!(n-p)!\}$ and $n!/\{(p-1)!\ (q-p-1)!(n-q)!\}$ by (n,p) and (n,p,q), respectively. The i-th moment of the p-th order statistic from the distribution is given by

$$E[X_{(p)}{}^i]=\alpha\beta^{-\alpha}(n,p)\sum_{k=0}^{p-1}(-1)^k\binom{p-1}{k}\int_0^\infty x^{\alpha+i-1}$$
$$\exp\{-n(-p+k+1)(x/\beta)^\alpha\}dx,$$
$$=i\beta^i\alpha^{-1}(n,p)\Gamma(i/\alpha)\sum_{k=0}^{p-1}(-1)^k\binom{p-1}{k}(n-p+k+1)^{-(1+i/\alpha)},$$

where $P=1,2,\cdots,n$, and $i=0,1,2,\cdots$. (2.1)

The product moment of the p-th and q-th order statistics from the distribution is given by

$$E[X_{(p)}X_{(q)}]=(\alpha/\beta^\alpha)^2(n,p,q)\sum_{k=0}^{p-1}\sum_{j=0}^{q-p-1}(-1)^{k+j}\binom{p-1}{k}\binom{q-p-1}{j}$$
$$\int_0^\infty x^\alpha\exp\{-(q-p-j+k)(x/\beta)^\alpha\}$$
$$\int_x^\infty y^\alpha\exp\{-(n-q+j+1)(y/\beta)^\alpha\}dydx. \qquad (2.2)$$

For convenience, let us denote the inside integral term in (2.2) by I_1. Let $t=(n-q+j+1)(y/\beta)^\alpha$. Then,

$$I_1=\alpha^{-1}a^{-(1+1/\alpha)}\Gamma(1+1/\alpha,ax^\alpha),$$

where $a=(n-q+j+1)/\beta^\alpha$ and $\Gamma(.,.)$ is incomplete gamma function.

Therefore, we can write(2.2) as follows;

$$E[X_{(p)}X_{(q)}]=\alpha\beta^{-2\alpha}(n,p,q)\sum_{k=0}^{p-1}\sum_{j=0}^{q-p-1}(-1)^{k+j}\binom{p-1}{k}\binom{q-p-1}{j}$$
$$a^{-(1+1/\alpha)}\int_0^\infty x^\alpha\exp\{-(q-p-j+k)(x/\beta)^\alpha\}$$
$$\Gamma(1+1/\alpha,\ ax^\alpha)dx \qquad (2.3)$$

Let us denote the integral term in (2.3) by I_2. Let $t=ax^\alpha$.

$$I_2=\alpha^{-1}a^{-(1+1/\alpha)}\int_0^\infty t^{1/\alpha}\exp(-bt)\Gamma(1+1/\alpha,t)dt,$$
$$=\alpha^{-1}a^{-(1+1/\alpha)}\Gamma(2+2/\alpha){}_2F_1(1,2+2/\alpha,\ 2+1/\alpha;\ b/(1+b))$$
$$/(1+1/\alpha)(1+b)^{2(1+1/\alpha)^2}$$

where $b=(q-p+j+k)/(n-q+j+1)$,

$_2F_1(a, b, c; z)$ is the hypergeometric function.

It follows that

$$E[X_{(p)}X_{(q)}]=\frac{2\beta^2}{\alpha}\frac{\alpha+2}{\alpha+1}\Gamma\left(\frac{2}{\alpha}\right)(n,p,q)\sum_{k=0}^{p-1}\sum_{j=0}^{q-p-1}(-1)^{k+j}\binom{p-1}{k}\binom{q-p-1}{j}(n-p+k+1)^{-2(1+1/\alpha)}\,{}_2F_1(1,2+2/\alpha,\ 2+1/\alpha;\ z), \tag{2.4}$$

where $z=(q-p+k-j)/(n-p+k+1)$.

The results(2.1) and (2.4) can be obtained by use of the results of Lieblein (1955) which are represented the gamma function and the incomplete beta function in the case of $\beta=1$. But the method shown in this section is simpler than that occured in Lieblein's paper, and the hypergeometric function is more useful than the incomplete beta function in evaluating the numerical value.

3. The Mean Square Error of the Quasi-range

It is well known that the quasi-range(including range) is an useful estimator as the measure of dispersion because of it's computational simplicity and the fact that censored data has rarely influence on it. Hence it is interesting problem that we find a good quasi-range as the estimator of the scale parameter β in the sense of mean square error for given shape parameter α and sample size n. The mean square error of the quasi-range is defined by the usual way, that is, $MSE(W_r)=E(W_r-\beta)^2$.

From the results (2.1) and (2.4), the first and second moments of the r-th quasi-range of random samples from the given distribution are given by

$$E[W_r]=\alpha^{-1}\beta\Gamma(1/\alpha)(n,r+1)\left\{\sum_{k=0}^{n-r-1}(-1)^k\binom{n-r-1}{k}(r+k+1)^{-(1+1/\alpha)}-\sum_{k=0}^{r}(-1)^r\binom{r}{k}(n-r+k)^{-(1+1/\alpha)}\right\} \tag{3.1}$$

and

$$E[W_r^2]=2\alpha^{-1}\beta^2\Gamma(2/\alpha)[(n,r+1)\left\{\sum_{k=0}^{n-r-1}(-1)^k\binom{n-r-1}{k}(r+k+1)^{-(1+2/\alpha)}+\sum_{k=0}^{r}(-1)^k\binom{r}{k}(n-r+k)^{-(1+2/\alpha)}\right\}$$

$$-\frac{2(\alpha+2)}{\alpha+1}(n,\ r+1,\ n-r)\sum_{k=0}^{r}\sum_{j=0}^{n-2r-2}(-1)^{k+j}\binom{r}{k}$$

$$\binom{n-2r-2}{j}(n-r+k)^{-2(1+1/\alpha)}{}_2F_1(1,\ 2+2/a,\ 2+1/\alpha;\ x)],$$

where $x=(n-2r-j+k-1)/(n-r+k)$. (3.2)

From (3.1) and (3.2), the variance and the mean square error of the r-th quasi-range of random samples from the distribution can be obtained.

Specifically if we take $\alpha=\beta=1$, then the given distribution becomes an exponential distribution and the mean and variance of its r-th quasi-range exactly agree with the results of Rider (1959).

The exact numerical values of the mean square errors for the quasi-ranges of random samples from the distribution will be evaluated by use of computer for $n=5$ (5)30, $r=0(1)$ $[(n-1)/2]$ and $\alpha=1/3, 1/2, 1,\ 2$.

Throughout the table 1, the sample quasi-ranges of the following cases are considered as good estimators in the sense of mean square error;

a) For a given shape parameter $\alpha=1/3$,
the 1st, 4th, 6th, 8th, 10th and 12th quasi-ranges are good estimators for the sample sizes 5, 10, 15, 20, 25 and 30, respectively.

b) For a given shape parameter $\alpha=1/2$,
the 1st, 4th, 7th, 9th and 11th quasi-ranges are good estimators for sample sizes 5, 10, 15, 20, 25 and 30, respectively.

c) For a given shape parameter $\alpha=1$,
the 1st, 2nd, 4th, 5th, 7th and 8th quasi-ranges are good estimators for sample sizes 5, 10, 15, 20, 25 and 30, respectively.

d) For a given shape parameter $\alpha=2$,
the 1st, 2nd, 3rd and 4th quasi-ranges are good estimators for the sample sizes 5, 10, 15 (or 20), 25 and 30, respectively.

4. Efficiencies of the Unbiased Estimators of the Scale Parameter Based on the Quasi-ranges.

We consider the efficiencies of the unbiased estimators of the scale parameter based on the r-th quasi-range ($r=0, 1, 2, \cdots,$ $[(n-1)/2]$.). The efficiency of the given estimator is defined as the ratio of the variance of the uniformly minimum variance unbiased estimator to the variance of the given estimator. Obviously the uniformly minimum variance unbiased estimator is more useful than any other estimator in the usual situations. But in view of the computational simplicity or for the censored data, the unbiased estimators based on the quasi-ranges may be more useful than any other estimators. Therefore it may be an interesting problem that we find a quasi-range based on an unbiased estimator of the scale parameter in distribution (1.1) which is most efficient among the r-th quasi-range ($r=0, 1, 2, \cdots[(n-1)/2].$).

Let

$$\tilde{\beta}=n^{1/\alpha}\Gamma(n)\hat{\beta}/\Gamma(n+1/\alpha)$$

where $\hat{\beta}=\left\{\frac{1}{n}\sum_{k=1}^{n}X_k^{\alpha}\right\}^{1/\alpha}$ is the maximum likelihood estimator of the scale parameter β in distribution (1.1) when the shape parameter α is known.

Then it can be shown that $\tilde{\beta}$ is the uniformly minimum variance unbiased estimator of the scale parameter β.

Since $\frac{1}{n}\sum_{k=1}^{n}X_k^{\alpha}$ has a gamma distribution with a shape parameter n and a scale parameter β^{α}/n, it can easily be shown that the 1st and 2nd moments of the maximum likelihood estimator of the scale parameter are given by

$$E[\hat{\beta}]=\Gamma(n+1/\alpha)\beta/\{n^{1/\alpha}\Gamma(n)\}$$

and

$$E[\hat{\beta}^2]=\Gamma(n+2/\alpha)\beta^2\{n^{2/\alpha}\Gamma(n)\}$$

It follow that

$$Var[\tilde{\beta}]=\{\Gamma(n)\Gamma(n+2/\alpha)/\Gamma^2(n+1/\alpha)-1\}\beta^2.$$

Now, we consider another unbiased estimator of the scale parameter based on the r-th quasi-range. Let $\beta_r=\beta W_r/E[W_r]$. Then β_r is clearly an unbiased estimator of the scale parameter β based on the r-th quasi-range. Moreover the variance of β_r can be obtained by the results in section 3. Therefore we obtain the efficiency of the unbiased estimator of the scale parameter based on the quasi-range, that is, $Eff[\beta_r]=Var[\tilde{\beta}]/Var[\beta_r]$.

The exact numerical values of the efficiencies (percent) of the unbiased estimators of the scale parameter based on the r-th quasi-range are evaluated for $r=0(1)[(n-1)/2]$, $n=5(5)30$ and $\alpha=1/3,\ 1/2,\ 1,\ 2$ by use of computer. Table 1 shows that an unbiased estimator of the scale parameter based on the quasi-range of random small sample from the distribution (1.1) is most efficient for the following case;

a) For a given shape parameter $\alpha=1/3$,
the estimator being based on the 2nd, 3rd, 4th, 5th and 6th quasi-ranges for the sample sizes $n=10$, 15, 20, 25, 30 respectively.

b) For a given shape parameter $\alpha=1/2$,
being based on the 1st, 2nd, 3rd, 4th and 5th quasi-ranges for the sample sizes $n=$ 10, 15, 20, 25, 30 respectively.

c) For a given shape parameter $\alpha=1$,
being based on the 1st, 2nd and 3rd quasi-ranges for the sample sizes $n=10$ (or 15), 20, 25(or 30), respectively.

d) For a given shape parameter $\alpha=2$,
being based on the range for the sample size $n=10$ or 15, and being based on the 1st quasi-range for the sample sizes $n=20$, 25 or 30.

5. The Mean Square Error of the Range and the Jackknife Estimator for the Range.

Table 1 shows that the range estimator for the scale parameter is worse than any other quasi-range estimator in the sense of mean square error except the case of the scale parameter $\alpha=2$. But the range may be more useful than any other quasi-range in view of computational simplicity. Hence we consider the jackknife estimator for the range defined by

$$J(W_0)=\frac{2n-1}{n}W_0-\frac{n-1}{n}W_1.$$

It is very interesting problem that we compare the relative mean square error of the range with that of the jackknife estimator for the range of random samples from the given distribution. As the mean square error of the range has been considered in section 2, we consider only the mean square error of the jackknife estimator for the range.

From the results (2.1) and (2.4), the first and second moments of the jackknife estimator for the range are represented as the following;

$$E[J(W_0)]=\frac{\beta}{\alpha}\Gamma(1/\alpha)[(2n-1)\Big\{\sum_{k=0}^{n-1}(-1)^k\binom{n-1}{k}(k+1)^{-(1+1/\alpha)}$$
$$-n^{-(1+1/\alpha)}\Big\}-(n-2)^2\sum_{K=0}^{n-2}(-1)^k\binom{n-2}{k}$$

$$(k-2)^{-(1+1/\alpha)}-(n-1)^{-(1+1/\alpha)}+n^{-(1+1/\alpha)}\}]$$

$$E[J(W_0)^2]$$

$$=\frac{2\beta^2}{\alpha}\Gamma(2/\alpha)\frac{(2n-1)^2}{n}\Big\{\sum_{k=0}^{n-1}(-1)^k\binom{n-1}{k}(k+1)^{-(1+2/\alpha)}$$

$$+n^{-(1+2/\alpha)}-\frac{2(\alpha+2)}{\alpha+1}(n-1)n^{-2(1+1/\alpha)}$$

$$\sum_{j=0}^{n-2}(-1)^j\binom{n-2}{j}{}_2F_1\Big(1,\ 2+2/\alpha,\ 2+1/\alpha;\ \frac{n-j-1}{n}\Big)$$

$$+\frac{2\beta^2}{\alpha}\Gamma(2/\alpha)\frac{(n-1)^3}{n}\sum_{k=0}^{n-2}(-1)^k\binom{n-2}{k}(k+2)^{-(1+2/\alpha)}$$

$$+(n-1)^{-(1+2/\alpha)}-n^{-(1+2/\alpha)}-\frac{2(\alpha+2)}{\alpha+1}(n-2)(n-3)$$

$$\Big\{(n-1)^{-2(1+1/\alpha)}\sum_{j=0}^{n-4}(-1)^3{}_2F_1\Big(1,\ 2+2/\alpha,\ 2+1/\alpha;\ \frac{n-j-3}{n-1}\Big)$$

$$+n^{-2(1+1/\alpha)}\sum_{j=0}^{n-4}(-1)^{j+1}\binom{n-4}{j}{}_2F_1\Big(1,\ 2+2/\alpha,\ 2+1/\alpha;\ \frac{n-j-2}{n}\Big)\Big\}\Big]$$

$$-\frac{4\beta^2}{\alpha}\ \frac{\alpha+2}{\alpha+1}\Gamma(2/\alpha)\frac{(2n-1)(n-2)^2}{n}\Big[n^{-2(1+1/\alpha)}$$

$${}_2F_1\Big(1,\ 2+2/\alpha,\ 2+1/\alpha;\ \frac{1}{n}\Big)-(n-2)\{n^{-2(1+1/\alpha)}$$

$$\sum_{j=0}^{n-3}(-1)^j\binom{n-3}{j}{}_2F_1\Big(1,\ 2+2/\alpha,\ 2+1/\alpha;\ \frac{n-j-2}{n}\Big)$$

$$+(n-1)^{-2(1+1/\alpha)}\sum_{j=0}^{n-3}(-1)^3\binom{n-3}{j}{}_2F_1\Big(1,\ 2+2/\alpha,\ 2+1/\alpha,\ \frac{n-j-2}{n-1}\Big)$$

$$+n^{-2(1+1/\alpha)}\sum_{j=0}^{n-3}(-1)^j\binom{n-3}{j}{}_2F_1\Big(1,\ 2+2/\alpha,\ 2+1/\alpha;\ \frac{n-j-1}{n}\Big)\Big\}$$

$$+(n-2)^{-2(1+1/\alpha)}\sum_{k=0}^{n-2}(-1)^k\binom{n-3}{k}{}_2F_1\Big(1,\ 2+2/\alpha,\ 2+1/\alpha;\ \frac{k+1}{k+2}\Big)\Big].$$

Therefore, we can obtain the mean square error of the jackknife estimator for the range of random samples from the distribution. The exact numerical values of the relative mean square errors of the jackknife estimator for the range are evaluated for the shape parameters $\alpha=1/3,\ 1/2,\ 1, 2$ and sample sizes $n=5(5)30$ by use of computer.

Table 3 shows that the jackknife estimator for the range in the sense of mean square error.

Table 1. The relative MSE of the quasi-ranges of random samples from the Weibull distribution.

n	r	$\alpha=1/3$	$\alpha=1/2$	$\alpha=1$	$\alpha=2$
5	0	3445.23679	93.97752	2.59721	0.16573
	1	82.36167	5.83408	0.38899	0.35955
10	0	6702.47052	175.45667	4.88490	0.29019
	1	324.73505	22.48375	1.04274	0.07994
	2	45.05807	4.80847	0.27042	0.19649
	3	7.63215	1.07430	0.27722	0.44613
	4	1.21999	0.64682	0.68000	0.79210
15	0	9807.56742	247.08161	6.64552	0.43551
	1	650.00992	41.58270	1.96360	0.09109
	2	123.54503	12.36341	0.67884	0.06167
	3	31.96784	4.15581	0.23171	0.14350
	4	9.04049	1.37019	0.15000	0.28958
	5	2.49629	0.50669	0.28261	0.48370
	6	0.82181	0.49745	0.57206	0.72047
20	0	12791.05180	311.97999	8.08464	0.56079
	1	1031.40236	61.50570	2.82624	0.14795
	2	232.99013	21.48431	1.22056	0.04824
	3	72.55874	8.82741	0.52289	0.05456
	4	25.77241	3.76766	0.21201	0.11744
	5	9.55778	1.56390	0.11340	0.21822
	6	3.49846	0.63222	0.15233	0.34848
	7	1.26496	0.34169	0.29294	0.50430
	8	0.61995	0.41913	0.51779	0.68422
	9	0.74100	0.74201	0.82000	0.88840
25	0	15672.32245	371.79504	9.31007	0.66778
	1	1452.25969	81.59637	3.61015	0.21284
	2	367.23089	31.47721	1.76853	0.07139
	3	128.26337	14.46249	0.89672	0.03430
	4	51.63126	7.09249	0.43729	0.05138
	5	22.23644	3.51541	0.20067	0.10394
	6	9.79549	1.69636	0.10161	0.18187
	7	4.27084	0.78332	0.09964	0.27809
	8	1.81258	0.37613	0.17005	0.38873
	9	0.79797	0.27336	0.30167	0.51671
	10	0.50517	0.37132	0.48512	0.66218
	11	0.61303	0.61945	0.71802	0.82303
30	0	18465.40555	427.56809	10.38241	0.76044
	1	1902.12714	101.56400	4.32383	0.27674
	2	523.34347	41.97254	2.29473	0.10778
	3	197.41268	20.73388	1.28799	0.04496
	4	86.25190	11.05832	0.72056	0.03071
	5	40.70684	6.07797	0.38609	0.04763
	6	19.97035	3.35130	0.19658	0.08913
	7	9.91880	1.79625	0.09888	0.15010
	8	4.88207	0.93332	0.07927	0.22742
	9	2.34541	0.46096	0.10839	0.31469
	10	1.10882	0.28148	0.18942	0.41697
	11	0.57277	0.24037	0.30730	0.52550
	12	0.43240	0.33875	0.46272	0.64957
	13	0.53284	0.53813	0.65061	0.78092
	14	0.80541	0.82445	0.87581	0.92414

Table 2. The efficiencies of the unbiased estimators of the scale parameter based on the r-th quasi-range of random samples from the Weibull distribution (%).

n	r	$\alpha=1/3$	$\alpha=1/2$	$\alpha=1$	$\alpha=2$
5	0	49.18	60.21	60.98	36.09
	1	52.75	54.49	38.46	15.18
10	0	35.50	46.20	51.98	36.83
	1	53.40	60.48	55.95	29.19
	2	56.42	59.20	45.62	19.11
	3	49.37	47.67	29.19	10.18
	4	27.92	22.90	10.00	2.96
15	0	29.18	38.55	44.72	34.48
	1	48.52	55.77	55.50	33.08
	2	56.61	61.28	54.40	30.45
	3	58.29	60.48	47.66	20.39
	4	55.65	55.19	37.79	14.18
	5	48.89	45.49	26.06	8.68
	6	35.58	29.50	13.27	3.94
20	0	25.32	33.59	39.49	32.02
	1	44.44	51.26	52.67	34.06
	2	54.15	59.12	55.68	30.68
	3	58.48	61.77	53.63	25.96
	4	59.50	61.13	48.62	21.03
	5	58.13	57.97	41.71	16.29
	6	54.67	52.51	33.53	11.91
	7	48.84	44.42	24.49	7.95
	8	39.24	32.63	14.90	4.44
	9	20.30	14.11	5.00	1.37
25	0	22.63	30.04	35.55	29.84
	1	41.14	47.47	49.64	33.95
	2	51.63	56.33	54.73	32.42
	3	57.02	60.61	55.34	29.13
	4	59.66	62.07	53.13	24.96
	5	60.32	61.53	48.91	19.81
	6	59.47	59.38	43.87	14.82
	7	57.36	55.79	37.66	11.39
	8	53.93	50.67	31.17	9.89
	9	48.89	43.79	23.46	7.91
	10	41.40	34.45	15.86	4.83
	11	29.98	21.10	7.99	2.22
30	0	20.61	27.33	32.47	27.95
	1	38.42	44.31	46.83	33.38
	2	48.95	53.62	53.17	32.92
	3	55.15	53.79	55.49	28.37
	4	58.70	61.41	54.42	26.47
	5	60.52	61.93	51.81	26.19
	6	60.92	61.25	47.42	24.98
	7	60.36	59.91	45.06	20.74
	8	58.91	56.64	38.74	16.03
	9	56.60	54.41	35.77	12.53
	10	53.35	46.31	29.14	9.32
	11	48.98	43.04	23.33	7.14
	12	42.84	35.80	17.19	4.18
	13	33.49	25.23	11.32	2.06
	14	16.36	10.34	3.32	0.95

Table 3. The relative MSE of the range and the jackknife estimator for the range of random samples from the Weibull distribution.

n	α	Rel. MSE(W_0)	Rel. MSE($J(W_0)$)
5	1/3	3445. 23679	10523. 91786
	1/2	93. 97752	271. 59691
	1	2. 59721	8. 14387
	2	0. 16537	3. 10665
10	1/3	6702. 11414	21452. 42965
	1/2	175. 45667	492. 78497
	1	4. 88490	12. 18505
	2	0. 29019	2. 94279
15	1/3	9807. 56743	31258. 85280
	1/2	247. 08161	669. 03545
	1	6. 64552	14. 90041
	2	0. 43551	2. 68386
20	1/3	12791. 05180	40360. 80879
	1/2	311. 97999	819. 02713
	1	8. 08464	16. 98882
	2	0. 56079	2. 53506
25	1/3	15672. 32245	48992. 47320
	1/2	371. 79504	951. 39015
	1	9. 31007	18. 70889
	2	0. 66778	2. 28654
30	1/3	18465. 40555	57293. 57706
	1/2	427. 56909	1071. 10816
	1	10. 38241	20. 18730
	2	0. 76045	2. 11787

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