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A Detection Matrix for 3^n Search Design

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ABSTRACT

A parallel flats fraction for the 3^n factorial experiment is defined as the union of flats, $\{t \mid \underline{A}t = \underline{C}_i \pmod 3\}$, $i=1,2,\cdots,f$, in $EG(n,3)$ and is symbolically written as $A\underline{t}=C$ where A is of rank r. The A matrix partitions the effects into $u+1$ alias sets where $u=(3^{n-r}-1)/2$. For each alias set the f flats produce an alias component permutation matrix (ACPM) with elements from S_3. In this paper, a detection vector of the ACPM was constructed for each combination of k or fewer two-factor interactions. Also the relationship between the detection vectors has been shown.

1. Introduction

A parallel flats fraction is defined as $T=\bigcup_{i=1}^{f}\{\underline{t};A\underline{t}=\underline{C}_i\}$ where A is an $r\times n$ matrix of rank r. Each equation $A\underline{t}=\underline{C}_i$ has 3^{n-r} points and is called a flat. The f flats have no points in common, hence are termed parallel, with $|T|=f\cdot 3^{n-r}$. The parallel flats fraction will be denoted symbolically by $A\underline{t}=C$, where $C=(\underline{C}_1,\underline{C}_2,\cdots,\underline{C}_f)$.

The choice of A determines the alias sets for the fraction. The estimate of an effect of the jth alias set from the ith flat, denoted by $\hat{S}_{ij}$, is actually a linear combination of all the factorial effects in that alias set. The form of the linear combination depends on $\underline{C}_i$, and is characterized by the permutation of levels of each effect in the set to the identified effect S_{ij}. These relations are given in the alias component permutation matrix (ACPM). The elements of the ACPM are from the permutation subgroup $\{e,(012),(021)\}$. The element of the ACPM for an effect E in the jth alias set for the ith flat can be computed from a single linear function of the elements of $\underline{C}_i$. The construction of a fraction is completed by the specification of A and C.

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2. Construction of a Detection Matrix

The general principle behind the detection procedure proposed here is the specification of certain linear combinations of the observations whose configuration of zero and nonzero values completely determines the non-negligible effects present.

We have defined $\hat{S}_{ij}$ as the estimate of the effect identified with the jth alias set for the ith flat where $i=1, 2, \cdots, f$ and $j=1, 2, \cdots, u$. These estimates are linear combinations of the 3^{n-r} observations on the flat.

The linear combinations of interest are $\hat{S}_{ij}-\hat{S}_{i'j}$, $i\neq i'=1, 2, \cdots, f$, $j=1, 2, \cdots, u$. The expected value of $S_{ij}-S_{i'j}$, is equal to zero in the following three cases:

(1) There are no nonzero effects in the alias set other than the one used to represent the set.

(2) There are nonzero effects in the alias set but their permutation relation to S_j is the same in flats i and i'.

(3) There are nonzero effects which have different permutation relations with S_j and just happen to combine to the same value in the two flats. The probability of this is assumed zero.

Now we construct a (0, 1) detection matrix K. The columns of K represent the differences (i.e., linear combinations) between the ith flat and the i' flat for each ACPM P_j, $j=1, 2, \cdots, u$. Since there are f flats and u ACPM, there are $\binom{f}{2}$ differences for each ACPM and hence $\binom{f}{2}\times u$ columns.

The rows of K represent all combinations of k or fewer two-factor interactions. The first row of K represents no interactions present. Since there are n main effects, there are $\binom{n}{2}$ two-factor interactions, say m. Therefore, the number of rows of K is

$$\binom{m}{0}+\binom{m}{1}+\binom{m}{2}+\binom{m}{3}+\cdots+\binom{m}{k}.$$

The elements of (0, 1) detection matrix K are determined by the following way. Note that each two-factor interaction represents four degrees of freedom.

For example, F_iF_j represents F_iF_j and $F_iF_j^2$, each with two degrees of freedom. The elements of each row will be called (0, 1) detection vector.

Suppose that a (0, 1) detection vector for the row F_iF_j is of interest. Then for each ACPM P_j the columns corresponding to main effects, F_iF_j and $F_iF_j^2$ are considered. But by the property of the design matrix A, F_iF_j and $F_iF_j^2$ belong to different alias sets. Therefore, for each ACPM P_j, main effects and F_iF_j or main effects and $F_iF_j^2$ are considered. If some alias sets do not contain F_iF_j or $F_iF_j^2$, then only columns corresponding to main effects are considered for that alias set.

In this way the submatrix with columns corresponding to main effects and F_iF_j or main effects and $F_iF_j^2$ or main effects are constructed from each ACPM. With these submatrices the differences between ith row and i' row (i.e., linear combination of rows of ACPM corresponding to the ith flat and i' flat) are checked. If the difference is zero then the corresponding element of a (0, 1) detection vector will be zero. If not, then the corresponding element will be one.

In general, the element K_{pq} of (0, 1) detection matrix can be determined by the following way:

$$k_{pq}=0 \quad \text{if } d_{ii'}=0$$
$$=1 \quad \text{if } d_{ii'}\neq 0,$$

where $p=1, 2, 3, \cdots, 1+m+\binom{m}{2}+\cdots+\binom{m}{k}$, $m=\binom{n}{2}$, and $q=1, 2, 3, \cdots, \binom{f}{2}\times u$, and $d_{ii'}$ is the difference between ith row and jth row of the submatrix obtained from ACPM. In this way, the (0, 1) detection vector for all combinations of two-factor interactions can be obtained.

3. Main Results

From the basic definition of the detection matrix the following lemmas are obtained directly.

Lemma 1. Suppose that two submatrices obtained from one ACPM for two combinations of interactions are identical. Then two submatrices produce the same (0, 1) detection elements for those two combinations.

Lemma 2. Suppose that two submatrices obtained from each ACPM for two combinations of interactions are identcal. Then (0, 1) detection vectors for two combinations of interactions are the same.

Lemma 3. Two (0, 1) detection vectors are distinct if and only if at least one of ACPM produce distinct (0, 1) detection vectors.

Theorem 1. Suppose that F_i and F_j, $i<j$, are aliased with each other in an alias set. Then for any given parallel flats fraction the (0, 1) detection vector for main

effects is identical with the (0, 1) detection vector for F_iF_j.

Proof. Suppose that F_i and F_j, $i<j$, are aliased with each other in the alias set S_k. It is noted that one of F_iF_j and $F_iF_j^2$ is in S_0 and the other is in S_k. Therefore, it is enough to show that for any given parallel flats fraction two submatrices, obtained from ACPM P_k, for main effects and for F_iF_j produce the same (0, 1) detection elements. Let $c_i=(c_1, c_2, \cdots, c_r)'$. Suppose that F_iF_j is contained in S_k. Then the following two cases are considered:

	F_i	F_j	F_iF_j	
(1)	0	c_l	$2c_l$	where $l\in\{1, 2, \cdots, r\}$
(2)	0	c_l+2c_m	$2c_l+c_m$	where $l\neq m\in\{1, 2, \cdots, r\}$.

The submatrix for main effects is composed of $(0, c_l)$ or $(0, c_l+2c_m)$ and the submatrix for F_iF_j is composed of $(0, c_l, 2c_l)$ or $(0, c_l+2c_m, 2c_l+c_m)$. Hence two submatrices produce the same (0, 1) detection elements for any given parallel flats fraction. Suppose that $F_iF_j^2$ are contained in S_k. Then the following two cases are considered:

	F_i	F_j	$F_iF_j^2$	
(1)	0	$2c_l$	c_l	where $l\in\{1, 2, \cdots, r\}$
(2)	0	$2c_l+2c_m$	c_m+c_l	where $l\neq m\in\{1, 2, \cdots, r\}$.

Similarly, two submatrices produce the same (0, 1) detection elements for any given parallel flats fraction.

Theorem 2. Suppose that F_i and F_j, $i<j$, are aliased with each other in an alias set. Then for any given parallel flats fraction the (0, 1) detection vector for F_mF_n, where $F_mF_n\neq F_iF_j$, is identical with the (0, 1) detection vector for (F_mF_n, F_iF_j).

Proof. Suppose that F_i and F_j, $i<j$, are aliased with each other in the alias set S_k, and F_mF_n and $F_mF_n^2$ are not contained in S_k. Then the following three cases can be considered:

(1) Alias sets which contain either F_mF_n or $F_mF_n^2$.

(2) S_k which contains F_iF_j or $F_iF_j^2$.

(3) Alias sets which do not contain F_mF_n or $F_mF_n^2$.

For each case two submatrices, obtained from ACPM, for F_mF_n and for (F_mF_n, F_iF_j) are the same for any given parallel flats. Therefore, for any given parallel flats, the (0, 1) detection vector for F_mF_n is identical with the (0, 1) detection vector for (F_mF_n, F_iF_j). Suppose that one of F_mF_n and $F_mF_n^2$ is contained in S_k, say F_mF_n. Then we consider the following three cases:

(1) Alias set which contains $F_mF_n^2$.

(2) S_k contains F_mF_n and F_iF_j or $F_iF_j^2$.

(3) Alias set which does not contain $F_mF_n^2$.

For each case two submatrices for F_mF_n and for (F_mF_n, F_iF_j) are the same for any given parallel flats. Hence the $(0, 1)$ detection vector for F_mF_n is identical with the $(0, 1)$ detection vector (F_mF_n, F_iF_j) for any given parallel flats.

Theorem 3. Suppose that F_i and F_j, $i<j$, are aliased with each other in the alias set S_k and F_a, F_b in S_r where $a<b$ and $k\neq r$. Then two $(0, 1)$ detection vectors for the following pairs are the same for any given parallel flats.

(1) main and F_iF_j

(2) main and F_aF_b

(3) main and (F_iF_j, F_aF_b)

(4) F_pF_q and (F_pF_q, F_iF_j) where $F_pF_q\neq F_iF_j$

(5) F_pF_q and (F_pF_q, F_aF_b) where $F_pF_q\neq F_aF_b$

Proof. Case 1, 2, 4 and 5 are immediately followed by Theorem 1 and Theorem 2. Suppose that S_k contains F_iF_j and S_r contains F_aF_b. Then $F_iF_j^2$ and $F_aF_b^2$ are contained in S_0. This means that no other alias set, except S_0, S_k and S_r, contains $F_iF_j, F_iF_j^2, F_aF_b$ and $F_aF_b^2$. Therefore, two submatrices, obtained from each ACPM (except S_k and S_r), for main effects and for (F_iF_j, F_aF_b) are the same for any given parallel flats. By Lemma 1 two submatrices produce the same $(0, 1)$ detection elements.

The submatrix obtained from P_k for (F_iF_j, F_aF_b) is exactly the submatrix obtained from P_k for F_iF_j. Therefore, two submatrices obtained from P_k for main effects and (F_iF_j, F_aF_b) produce the same $(0, 1)$ detection elements by case 1. Similarly two submatrices obtained from P_r produce the same $(0, 1)$ detection elements by case 2.

The above arguments hold for the following cases:

(1) S_k contains F_iF_j and S_r contains $F_aF_b^2$.

(2) S_k contains $F_iF_j^2$ and S_r contains F_aF_b.

(3) S_k contains $F_iF_j^2$ and S_r contains $F_aF_b^2$.

Hence case 3 is proved.

Theorem 3 implies that $(0, 1)$ detection vectors for F_iF_j, for F_aF_b and for (F_iF_j, F_aF_b) are identcal, and $(0, 1)$ detection vectors for (F_pF_q, F_iF_j) and for (F_pF_q, F_aF_b)

are indentical for any given parallel flats if F_i and F_j are aliased in S_k and F_a, F_b in S_r.

4. Example

Consider a 3^4 factorial experiment for which it can be assumed that all three and four-factor interaction effects are negligible. The A matrix for this example will be taken as

$$A=\begin{bmatrix}1 & 1 & 1 & 0\\ 1 & 2 & 0 & 1\end{bmatrix},$$

thus there are flats of size nine. The alias sets are

$$S_0=\{u\},$$
$$S_1=\{F_1, F_2F_3, F_2F_4^2, F_3F_4\},\quad S_2=\{F_2, F_1F_3, F_1F_4, F_3F_4^2\},$$
$$S_3=\{F_3, F_1F_2, F_1F_4^2, F_2F_4\},\quad S_4=\{F_4, F_1F_2^2, F_1F_3^2, F_2F_3^2\}.$$

An example of a parallel flats fraction in 27 runs is given with

$$C=(\underline{C}_1, \underline{C}_2, \underline{C}_3) \text{ as } C=\begin{bmatrix}0 & 0 & 1\\ 0 & 1 & 2\end{bmatrix}.$$

By choosing the main effect in each alias set as the identified effect, the ACPM are

$$\begin{array}{cccc} F_1 & F_2F_3 & F_2F_4^2 & F_3F_4 \end{array}\qquad\qquad \begin{array}{cccc} F_2 & F_1F_3 & F_1F_4 & F_3F_4^2 \end{array}$$

$$P_1=\begin{bmatrix} e & e & e & e\\ e & e & (021) & (012)\\ e & (021) & (012) & e \end{bmatrix}\qquad P_2=\begin{bmatrix} e & e & e & e\\ e & e & (012) & (021)\\ e & (021) & (021) & (021) \end{bmatrix}$$

$$\begin{array}{cccc} F_3 & F_1F_2 & F_1F_4^2 & F_2F_4 \end{array}\qquad\qquad \begin{array}{cccc} F_4 & F_1F_2^2 & F_1F_3^2 & F_2F_3^2 \end{array}$$

$$P_3=\begin{bmatrix} e & e & e & e\\ e & e & (021) & (012)\\ e & (021) & e & (012) \end{bmatrix}\qquad P_4=\begin{bmatrix} e & e & e & e\\ e & (021) & (021) & (021)\\ e & (012) & e & (021) \end{bmatrix}$$

For each ACPM the first column consists of e.

Then clearly the elements of the first row of K are zero. Consider two factor interaction F_1F_3. Since alias sets S_1 and S_3 do not have the two factor interaction F_1F_3, it is enough to consider the main effect column of ACPM P_1 and P_3, hence the corresponding detection elements are (0 0 0), (0 0 0) respectively. Since F_1F_3 is contained in S_2, the submatrix obtained from P_2 is

$$\begin{array}{cc} F_2 & F_1F_3 \end{array}$$
$$\begin{bmatrix} e & e \\ e & e \\ e & (021) \end{bmatrix}$$

The corresponding detection elements are (0 1 1). Similarly, the detection elements (1 0 1) are obtained from P_4. Hence the (0, 1) detection vector for row F_1F_3 is(0 0 0 0 1 1 0 0 0 1 0 1). In a similar way all (0, 1) detetion vectors can be obtained.

Table 1 shows the (0, 1) detection matrix for 3^4 factorial obtained with $C=\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 2 \end{bmatrix}$. The first row indicates the ACPM matrices, and the second row the difference between the ith row and the i' row of ACPM matrices. The first column denotes the number (0, 1) detection vectors and the next four columns the subscript of two-factor interactions.

Table 1. The (0, 1) Detection Matrix for the 3^4 Factorial

			P_1			P_2			P_3			P_4	
		1—2	1—3	2—3	1—2	1—3	2—3	1—2	1—3	2—3	1—2	1—3	2—3
1	MAIN	0	0	0	0	0	0	0	0	0	0	0	0
2	12	0	0	0	0	0	0	0	1	1	1	1	1
3	13	0	0	0	0	1	1	0	0	0	1	0	1
4	14	0	0	0	1	1	1	1	0	1	0	0	0
5	23	0	1	1	0	0	0	0	0	0	1	1	0
6	24	1	1	1	0	0	0	1	1	0	0	0	0
7	34	1	0	1	1	1	0	0	0	0	0	0	0
8	12 13	0	0	0	0	1	1	0	1	1	1	1	1
9	12 14	0	0	0	1	1	1	1	1	1	1	1	1
10	12 23	0	1	1	0	0	0	0	1	1	1	1	1
11	12 22	1	1	1	0	0	0	1	1	1	1	1	1
12	12 34	1	0	1	1	1	0	0	1	1	1	1	1
13	13 14	0	0	0	1	1	1	1	0	1	1	0	1
14	13 23	0	1	1	0	1	1	0	0	0	1	1	1
15	13 24	1	1	1	0	1	1	1	1	0	1	0	1
16	13 34	1	0	1	1	1	1	0	0	0	1	0	1
17	14 23	0	1	1	1	1	1	1	0	1	1	1	0
18	14 24	1	1	1	1	1	1	1	1	1	0	0	0
19	14 34	1	0	1	1	1	1	1	0	1	0	0	0
20	23 24	1	1	1	0	0	0	1	1	0	1	1	0
21	23 34	1	1	1	1	1	0	0	0	0	1	1	0
22	24 34	1	1	1	1	1	0	1	1	0	0	0	0

column 4—8 denote the subscripts of two-factor interactions

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