선형연속데이터형제어계통의 플랜트와 디지탈모델의 오차자승적분지표에 의한 최적디지탈제어기의 전달함수 유도

論 文
32～6～4

Deriviation of the z-transfer Function of Optimal Digital Controller Using an Integral-Square-Error Criterion with the Continuous-data Model in Linear Control Systems

崔 京 三*
(Kyung-Sam Choi)

요 약

본 논문은 선형연속데이터형제어계통을 디지탈화하는 경우에 연속시스템과 디지탈시스템의 연속상태궤도가 정합하도록 아날로그 제어기를 대찬 할 수 있는 최적디지탈 제어기의 전달함수를 유도한 것이다. 간단한 수치계산예를 통하여 본 방법이 유효함을 알 수 있었다.

Abstract

In this paper, an attempt is made to match the continuous state trajectory of the digital control system with that of its continuous data model. Matching the state trajectories instead of the output responses assures that the performances of the internal variables of the plant as well as the output variables are preserved in the discretization.

The mathematical tool used in this research is an extended maximum principle of the Pontryagin type, which enables one to synthesize a staircase type of optimal control signals, such as the output signal of a zero-order hold associated with a digital controller.

A general mathematical expression of the digital controller which may be used to replace the analog controller of a general system while preserving as mauch as possible the performance characteristics of the original continuous-data control system is derived in this paper.

1. Introduction

Digital approximation of continuous-data control systems has been of increasing interest to industrial control, navigation, and flight control sytems because sampled-data and digital control have more advantages (less effect due to noise and disturbance, no drift, improved sensitivity, better reliability, more compact and lightweight, and less cost and so

* 正 會 員 : 弘益大 工大 電氣工學科 敎授 · 工博
接受日字 : 1983年 3月 2日

forth) than continuous control[1]. From the standpoint of implementation, the sampling rate should be sufficiently low in order to allow time for computation and for the computer to be time-shared. But lower limits of the sampling rate are determined by factors such as roughness in the time response, errors due to measurement noise and sensitivity to plant parameter uncertainty and disturbances. The control system designer is then faced with the task of optimizing the performance of the digital control system at given rate of sampling.

In converting a contrinuous-data controller into a digital controller, ad hoc approaches such as prewarped bilnear transform and Tustin transform techniques have typically been used. These methods have the advantage of being straightforward and easy to use, and they are intuitively appealing. But the performance of a system digitalized by these approaches resembles the performance of the continuous system only when the sampling frequency is relatively high, because the dynamics of the plant and the feedback structure of the system are not taken into considerations.

Five years ago Rattan[2] presented a method using a complex-curve fitting technique to synthesize the ditigal controller so that the frequency response of the digitalized system matches that of the original continuous model with a least-square fit. This method is better than Tustin transform approach, especially for lower sampling frequencies[3]. However, this method does not take the time-domain performances into consideration; and only the magnitude plots of the frequency responses are matched, without regard to the phase plots. To compensate these shortcomings, the state-variable design techniques in time-domain should be developed.

In this study, an attempt is made to match the continuous state trajectory of the digital control system with that of its analog (continuous-data) model. Matching the state trajectories instead of the output responses assures that the performances of the internal variables of the plant as well as the output variable are preserved in the discretization. It should also be emphasized that the matching is specified over the entire continuous time axis, not just at discrete sampling instants, and is quantified by a minimum integral squared error. The choice of this performance criterion is motivated by the fact that if the state trajectories of two linear dynamical systems match, then frequency responses of the two systems will also match, as seen by Laplace-transformation of the state equations.

The mathematical tool used in this research is an extended maximum principle of the Pontryagin type, which enables one to synthesize a "staircase" type of optimal control signals, such as the output signal of a zero-order hold associated with a digital controller. The extended maximum principle was initiated by Chang[4] and further developed by Yeh and his co-workers[5)-8)].

2. Objectives

The main objective of this research is to derive a general mathematical expression of the digital controller which may be used to replace the continuous controller of a general system while preserving as much as possible the performance characteristics of the original continuous system. The specific objectives are

1) To derive an optimal control law for the digital control system.
2) To derive the z-transfer function of the optimal digital controller in terms of the parameters of the continuous model.

No attempt has been made, however, to obtain numerical results for control examples as a comparative study, nor was there an attempt to develop computer programs for evalution of the z-transfer function of the digital controller, as these will be proposed for further research.

3. Formulation of the Design Problem

Consider a linear continuous-data control system (Fig. 1) that has satisfactory (or ideal) performances. The state and output equations of the plant are given by

$$\dot{x}_a = Ax_a(t) + bu_m(t) \tag{1}$$

$$y_m(t) = c'x_a(t) + du_m(t) \tag{2}$$

The state and output equations of the controller are given by

$$\begin{cases} \dot{x}_c(t) = A_c x_c(t) + b_c e_m(t) & (3) \\ u_m(t) = c_c' x_c(t) + d_c e_m(t) & (4) \end{cases}$$

Where $x_a(t)$ and $x_c(t)$ are n and n_c dimensional vectors, respectively, $u_m(t)$, $e_m(t)$, $y_m(t)$ and $r(t)$ are scalar functions. The dimensions of the coefficient matrices are commensurate with the vectors with which they associate.

The design objective is to replace the controller $G_c(s)$ by a digital controller $D(z)$ such that the state trajectory of the digitalized system matches that of the continuous model as closely as possible. The digital control system is represented by Fig. 2, where $G(s)$ is the same plant as in the continuous model, and $D(z)$ is to be synthesized in such a way that when $r(t)$ is a step function, the performance index

$$J = \int_0^\infty \frac{1}{2}\Big\{ [x(t) - x_a(t)]' Q[x(t) - x_a(t)] + \beta[u(t) - u_m(t)]^2 \Big\} dt \tag{5}$$

attains its minimum, where

$$Q = \begin{bmatrix} q_1 & 0 & \cdots & 0 \\ 0 & q_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & q_n \end{bmatrix} \tag{6}$$

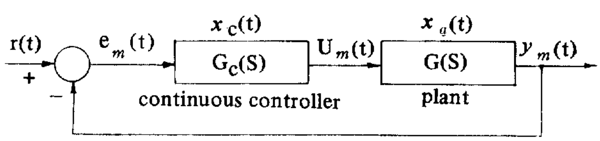

Fig. 1. Linear continuous system model.

Note that the performance index is an integral, not a discrete sum. Therefore attempt is made to match the trajectories over continuous time axis, not just at the sampling instants. The state and output equations of the plant in the digital control system are

$$\begin{cases} \dot{x}(t) = Ax(t) + bu(kT) & (7) \\ y(t) = c'x(t) + du(kT) & (8) \end{cases}$$

for $kT \le t < (k+1)T$, on account of the zero-order hold used in the digital control system (Fig. 2).

4. The Optimal Strategy

4.1 The Extended Maximum Principle

An extended version of the maximum principle of Pontryagin will be used to find the optimal control sequence $u(kT)$, $k=1,2,\ldots$, which minimizes the performance index[5]. The error sequence $e(kT)$ can be expressed in terms of $u(kt)$; and the digital controller $D(z)$ can be determined by

$$D(z) = \frac{U(z)}{E(z)} \tag{9}$$

where $U(z)$ is the z-transform of $u(kT)$ and $E(z)$ is the z-transform of $e(kT)$.

The extended maximum principle may be applied to the case where the control inputs are outputs of zero-order holds[5)-9)]. It can be derived that a necessary condition for an admissible control $u(t)$ to be optimal is that

$$\int_{kT}^{(k+1)T} \Big\{ b'p(t) - \beta[u(kT) - u_m(t)] \Big\} dt = 0 \tag{10}$$

where $p(t)$ is the state vector of the adjoint system satisfying

$$\dot{p}(t) = -\frac{\partial H[x(t), p(t), u(t)]}{\partial x(t)} \tag{11}$$

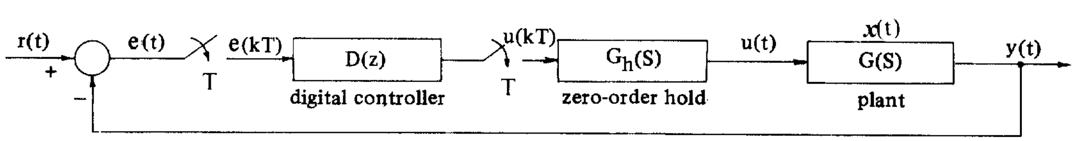

Fig. 2. The digitalized system.

and $H[x(t), p(t), u(t)]$ is the Hamiltonian function given by

$$H[x(t), p(t), u(t)] = p'(t)\dot{x}(t) - \frac{1}{2}\left\{[x(t)-x_a(t)]' \cdot Q[x(t)-x_a(t)] + \beta[u(t)-u_m(t)]^2\right\} \quad (12)$$

Now substituting Eq. (12) into Eq. (11) and invoking Eq. (7) gives the adjoint state equation

$$\dot{p}(t) = -A'p(t) + Q[x(t)-x_a(t)] \quad (13)$$

4.2 Determination of Optimal Control Sequence

In order to solve Eq. (10) for the optimal control sequence, the solution $p(t)$ must first be obtained from Eq. (13), which calls for the solutions of $x(t)$ in terms of $u(kT)$, and $x_a(t)$ in terms of $r(t)$.

Let the augmented state vector of the model be

$$x_m(t) = \begin{bmatrix} x_a(t) \\ \cdots\cdots \\ x_c(t) \end{bmatrix} \quad (14)$$

Then

$$\dot{x}_m(t) = A_m x_m(t) + b_m r(t) \quad (15)$$

where

$$A_m = \begin{bmatrix} A - \dfrac{bd_c c'}{1+dd_c} & \dfrac{bc'_c}{1+dd_c} \\ \dfrac{-b_c c'}{1+dd_c} & A_c - \dfrac{b_c d c_c}{1+dd_c} \end{bmatrix} \quad (16)$$

$$b_m = \begin{bmatrix} \dfrac{bd_c}{1+dd_c} \\ \dfrac{b_c}{1+dd_c} \end{bmatrix} \quad (17)$$

Define Q_m to be the $n \times (n+n)$ matrix obtained by augmenting columns of zeros to Q, *i.e.*,

$$Q_m = [Q \vdots 0] \quad (18)$$

Then by definition

$$Qx_a(t) = Q_m x_m(t) \quad (19)$$

Assume that $r(t)$ is a step function, *i.e.*,

$$r(t) = \begin{cases} \alpha & t \geqq 0 \\ 0 & t < 0 \end{cases} \quad (20)$$

Solutions of the state and adjoint equations, $u_m(t)$ become

$$x(t) = \Phi(t-kT)\, x(kT) + \Phi^s(t-kT)\, bu(kT) \quad (21)$$

$$x_m(t) = \Phi_m(t-kT)\, x_m(kT) + \Phi^s_m(t-kT) b_m \alpha \quad (22)$$

$$p(t) = \psi(t-kT)\, p(kT) + F(t-kT)\, x(kT) + F^s(t-kT)\, bu(kT) - F_m(t-kT)\, x_m(kT) - F^s_m(t-kT)\, b_m \alpha \quad (23)$$

$$u_m(t) = c'_m x_m(t) + d_m \alpha \quad (24)$$

for $kT \leqq t < (k+1)T$.

where

$$\Phi(t) = \epsilon^{At} \quad (25)$$

$$\Phi^s(t) = \int_0^t \Phi(\tau)\, d\tau \quad (26)$$

$$\Phi_m(t) = \epsilon^{A_m t} \quad (27)$$

$$\Phi^s_m(t) = \int_0^t \Phi_m(\tau) d\tau \quad (28)$$

$$c'_m = [-d_c c'/1+dd_c \quad c'_c/1+dd_c] \quad (29)$$

$$d_m = d_c/1+dd_c \quad (30)$$

$$\Psi(t) = \epsilon^{-A't} \quad (31)$$

$$F(t) = \int_0^t \Psi(t-\tau)\, Q\, \Phi(\tau)\, d\tau \quad (32)$$

$$F^S(t) = \int_0^t F(\tau)\, d\tau \quad (33)$$

$$F_m(t) = \int_0^t \Psi(t-\tau)\, Q_m\, \Psi_m(\tau)\, d\tau \quad (34)$$

$$F^S_m(t) = \int_0^t F_m(\tau)\, d\tau \quad (35)$$

Substituting Eq. (23) & Eq. (24) into Eq. (10) gives

$$\beta Tu(kT) = b'[\psi^s(T)\,p(kT) + F^s(T)\,x(kT) + F^{ss}(T)bu(kT) - F^s_m(T)\,x_m(kT) - F^{ss}_m(T)\,b_m\alpha] + \beta c'_m[\Phi^s_m(T)\,x_m(kT) + \Phi^{ss}_m(T)\,b_m\alpha] + \beta Td_m\alpha \quad (36)$$

where

$$\psi^s(T) = \int_0^T \psi(t)\,dt \quad (37)$$

$$F^{ss}(T) = \int_0^T F^S(t)\,dt \quad (38)$$

$$F^{ss}_m(T) = \int_0^T F^S_m(t)\,dt \quad (39)$$

$$\Phi^{ss}_m(t) = \int_0^T \Phi^S_m(t)\,dt \quad (40)$$

Solving Eq. (36) for the optimal control sequence gives

$$u(kT) = \Big\{[b'F^s_m(T) - \beta c'_m\,\Phi^s_m(T)\,x_m(kT) - b'[F^S(T)\,x(kT) + \psi^s(T)\,p(kT)] + [(b'F^{ss}_m(T) - \beta c'_m\,\Phi^{ss}_m(T))\,b_m - \beta Td_m]\,\alpha\Big\} / [b'F^{ss}(T)\,b - \beta T] \quad (41)$$

5. The Digital Controller

5.1 z-transform of the Optimal Control Sequence

In order to determine $U(z)$ for use in Eq(9), $X(z)$, $X_m(z)$, and $P(z)$ must be determined first. Setting $t=(k+1)T$ in Eqs. (21)~(23) gives

$$x[(k+1)T] = \Phi(T)\,x(kT) + \Phi^s\,bu(kT) \quad (42)$$

$$x_m[(k+1)T] = \Phi_m(T)\,x_m(kT) + \Phi^s_m(T)\,b_m\alpha \quad (43)$$

$$p[(k+1)T] = \psi(T)\,p(kT) + F(T)\,x(kT) - F_m(T)\,x_m(kT) + F^s(T)\,bu(kT) - F^s_m(T)\,b_m\alpha \quad (44)$$

Taking z-transformation of Eqs. (42)~(44) gives, respectively

$$x(z) = \hat{\Phi}(z)\,[zx(0) + \Phi^s(T)\,bu(z)] \quad (45)$$

$$x_m(z) = \hat{\Phi}_m(z)\,[zx_m(0) + \Phi^s_m(T)\,b_m\,\alpha z/(z-1)] \quad (46)$$

$$p(z) = \hat{\psi}(z)\,[zp(0) + F(T)\,x(z) - F_m(T)\,x_m(z) + F^s(T)\,bu(z) - F^s_m(T)\,b_m\,\alpha z/(z-1)] \quad (47)$$

where

$$\hat{\Phi}_m(z) = [zI - \Phi(t)]^{-1} \quad (48)$$

$$\hat{\Phi}_m(z) = [zI - \Phi_m(T)]^{-1} \quad (49)$$

$$\hat{\psi}(z) = [zI - \psi(T)]^{-1} \quad (50)$$

Substituting Eqs. (45) and (46) into Eqs. (47) gives

$$p(z) = \hat{\psi}(z)\Big\{zp(0) + F(T)\,\hat{\Phi}(z)\,zx(0) - F_m(T)\,\hat{\Phi}_m(z)zx_m(0) - [F_m(T)\,\hat{\Phi}_m(z)\,\Phi^s_m(T) + F^s_m(T)]\,b_m\,\alpha z/(z-1) + [F(T)\,\hat{\Phi}(z)\,\Phi^s(T) + F^s(T)]\,bU(z) \quad (51)$$

Substituting Eqs. (45), (46), and (51) into the z-transform of Eq. (41), we obtain a solution $U(z)$ as

$$U(z) = [b'K_m(z) - \beta c'_m\,\Phi^s_m(T)\,\hat{\Phi}_m(z)]\,z\,x_m(0) - b'[K_0(z)z\,p(0) + K(z)\,zx(0)] + [b'H_m(z)b_m - \beta c'_m\,H_0(z)\,b_m - \beta Td_m]\,\alpha z/(z-1)\,/[b'H(z)\,b - \beta T] \quad (52)$$

where

$$K_0(z) = \psi^s(T)\,\hat{\psi}(z) \quad (53)$$

$$K(z) = [F^s(T) + K_0(z)\,F(T)]\,\hat{\Phi}(z) \quad (54)$$

$$K_m(z) = [F^s_m(T) + K_0(z)\,F_m(T)]\,\hat{\Phi}_m(z) \quad (55)$$

$$H_0(z) = \Phi^{ss}_m(T) + \Phi^S_m(T)\hat{\Phi}_m(z)\,\Phi^S_m(T) \quad (56)$$

$$H(z) = F^{ss}(T) + K_0(z)\,F^s(T) + K(z)\,\Phi^s(T) \quad (57)$$

$$H_m(z) = F^{ss}_m(T) + K_0(z)\,F^s_m(T) + K_m(z)\,\Phi^s_m(T) \quad (58)$$

5.2 Z-transform of the Error

From the block diagram of Fig. 2 and Eqs. (8) and (21) we may write

$$e(t)=\alpha-y(t) \tag{59}$$

$$y(t)=c'\Phi(t-kT)x(kT)+[c'\ \Phi^{s}(t-kT)b+d]\ u(kT) \tag{60}$$

for $kT\leqslant t<(k+1)T$. Therefore

$$e[(k+1)T]=\alpha-c\ \Phi(T)\,x(kT)-[c'\Phi^{s}(T)\,b+d]\,u(kT) \tag{61}$$

Taking z-transform of the above equation, we have

$$E(z)=e(0^{+})+\alpha/(z-1)-c'\Phi(T)\,x(z)\,z^{-1}-[c'\Phi^{s}(T)b+d]\ z^{-1}\ U(z) \tag{62}$$

where $e(0^{+})$ is found from Eq. (59) and Eq. (8).

$$e(0^{+})=\alpha-c'x(0^{+})-du(0^{+}) \tag{63}$$

Substituting Eqs. (63) and (45) into Eq. (62) we have

$$E(z)=\alpha z/(z-1)-du(0^{+})-c'[\mathrm{I}+\Phi(T)\,\hat{\Phi}(z)]\ x(0)-z^{-1}\left\{c'[\mathrm{I}+\Phi(T)\,\hat{\Phi}(z)]\ \Phi^{s}(T)b+d\right\}U(z) \tag{64}$$

where $u(0^{+})$ may be obtained from Eq. (41). Setting k=0 in Eq. (41) and substituting the resulting expression of $u(0^{+})$ into Eq. (64) yields

$$\begin{aligned}E(z)=&\,db'\ \psi^{s}(T)\,p(0)/[b'F^{ss}(T)\ b'-\beta T]\\&+\{db'F^{s}\ (T)\,|\,[b'F_{ss}(T)\,b-\beta T]-c'[\mathrm{I}+\Phi(T)\hat{\Phi}\\&(z)]\}\ x(0)-d[b'F^{s}_{m}(T)-\beta c'_{m}\ \Phi^{s}(T)]\ x_{m}(0)\\&-z^{-1}\ \{c'[\mathrm{I}+\Phi(T)\ \hat{\Phi}(z)]\ \Phi^{s}(T)b+d\}\ U(z)\\&+\{z/(z-1)-d[(b'F^{ss}_{m}(T)-\beta c'_{m}\ \Phi^{ss}_{m}(T))\ b_{m}\\&-\beta Td_{m}]/[b'F^{ss}(T)\ b-\beta T]\}\ \alpha\end{aligned} \tag{65}$$

5.3 The z-transfer Function of the Digital Controller

By virtue of the relationship given in Eq. (9), the z-transfer function of the digital controller may now be written, provided that the initial conditions $x_{m}(0), x(0)$ and $p(0)$ are known.

For the system under consideration, the state trajectory of the continuous model is assumed to start from $x_{m}(0)=\mathbf{0}$. Let $x(0)$, the initial state of the plant of the digital control system, be unspecified. Then $p(0)=\mathbf{0}$. However, since the minimization of the performance index given in Eq. (5) means the continuous matching of $x(t)$ with $x_{a}(t)$ over an infinitely long period of time, it is reasonable to conjecture that $x(t)$ starts at the same point as $x_{a}(t)$, or very close to it, provided that the sampling frequency is considerably higher than the natural frequency of the control system. Hence the initial condition may be chosen as

$$x_{m}(0)=\mathbf{0} \tag{66}$$

$$x(0)=\mathbf{0} \tag{67}$$

$$\mathrm{P}(0)=\mathbf{0} \tag{68}$$

Substituting these conditions into Eqs. (52) and (65) and using the resulting expressions in Eq. (9), we obtain

$$\begin{aligned}D(z)^{-1}=&\,[b'H(z)b-\beta T]\ \{\ z-d(z-1)\ [(b'F^{ss}_{m}(T)\\&-\beta c'_{m}\ \Phi^{ss}_{m}(T))\ b_{m}-\beta Td_{m}]/[b'F^{ss}(T)b\\&-\beta T]\}/[b'H_{m}(z)b_{m}-\beta c'_{m}H_{0}(z)b_{m}\\&-\beta Td_{m}]\ z-\{c'[\mathrm{I}+\Phi(T)\hat{\Phi}(z)]\,\Phi^{s}(T)b\\&+d\}\ z^{-1}\end{aligned} \tag{69}$$

For most control systems, there is no direct linkage between the control signal and the output. For these systems the coefficient d is zero, and the z-transfer function of the digital controller is

$$D(z)=\{[b'H(z)b-\beta T]/[b'H_{m}(z)b_{m}-\beta c'_{m}H_{0}(z)b_{m}-\beta Td_{m}]-c'\ \hat{\Phi}(z)\ \Phi^{s}(T)b\}^{-1} \tag{70}$$

6. Illustrative Example

In this section a simple continuous system with unity feedback is digitalized by the method given in this paper. The block diagram of the illustrative continuous-data control system is as shown in Fig. 3.

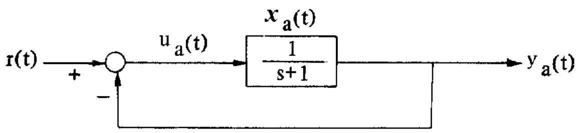

Fig. 3. A simple continuous system.

The problem is to discretize, i.e. to design a digital controller as shown in Fig. 2 so that the state trajectory of the continuous-data system matches that of the digitalized system.

It is seen from Fig. 3, Eqs. (16), (17), (29), and Eq. (30) that

$$G(s) = 1/(s+1)$$

$$A_c = [0],\quad \boldsymbol{b}_c = [0],\quad \boldsymbol{c}'_c = [0],\quad d_c = [1]$$

$$A = [-1],\quad \boldsymbol{b} = [1],\quad \boldsymbol{c} = [1],\quad d = [0]$$

$$A_m = [-2],\quad \boldsymbol{b}_m = [1],\quad \boldsymbol{c}_m = [-1],\quad d_m = [1]$$

Therefore, the z-transfer function of the optimal digital controller under the conditions that $\alpha=1$, $\beta=1$, and $Q=Q_m=1$ is given by

$$D(z) = \frac{0.980520(z-0.945007)(z-0.960789)}{(z-0.945018)(z-0.961544)}$$

Table 1. Values of $y_a(t)$ and y(t).

No.	t	$y_a(t)$	y(t)
1	0.00	0.00000	0.00000
2	0.04	0.03844	0.03845
3	0.08	0.07393	0.07394
4	0.20	0.16484	0.16486
5	0.40	0.27534	0.27536
6	0.60	0.34940	0.34943
7	0.80	0.39905	0.39908
8	1.00	0.43233	0.43235
9	1.60	0.47962	0.47963
10	2.60	0.49724	0.49724
11	3.60	0.49963	0.49962

When the sampling period T is selected to be 0.04 [sec] the outputs of the continuous-data system and the digitalized system with the optimal digital controller are shown in Table 1.

7. Concluding Remarks

A general mathematical expression of the optimal digital controller which can be used to replace the analog controller of a continuous data control system while matching the continuous state trajectory of the digitalized system with that of its continuous-data model was derived in this paper by using an extended maximum principle of the Pontryagin type. Matching the state trajectories instead of the output responses assures that the performances of the internal variables of the plant as well as the output variables are preserved in the discretization. The author emphasize that the matching is specified over the entire continuous time axis, not just at discrete sampling instants, and is quantified by a minimum integral squared error.

It is seen from a simple illustrative numerical example that the deriviation of the z-transfer function of optimal digital controller developed in this study can be effectively used to design optimal digital controller. And the digitalized control system will be run under less effect due to noise and disturbance, no drift, improved sensitivity, and better reliability.

Obtaining numerical results for control examples as a comparative study and developing computer programs for evaluation of the z-transfer function of the optimal digital controller are proposed for further research.

(Acknowledgement)

The author would like to thank Dr. Yeh, the professor of the department of electrical engineering, in the University of Kentucky for his kind cooperation in this research. The research reported in this paper was partially supported by the Asan Social Welfare Foundation.

References

1) Kuo, B.C.; Digital control systems, Holt, Rinehart and Winston, 1980, pp. 1-5.
2) Rattan, K.S. and H.H. Yeh; "Discretizing continuous-data control systems", *Computer Aided Design*, vol. 10, No. 5, pp. 299-306, 1978.
3) Tattan, K.S.; "Digitalization of existing continuous-data control systems", AFWAL-TM-80-105-FIGC, September 1980, Flight Dynamics Laboratory, Wright-Patterson Air Force Base, Ohio.
4) Chang, S.S.L.; "A general theory of optimal processes", *SIAM Journal of Control, Series A*, vol. 4, pp. 46-54, 1966.
5) Yeh, H.H. and J.T. Tou; "On the general theory of optimal processes", *International Journal of Control*, vol. 9, No. 4, pp. 433-451, 1969.
6) Yeh, H.H. and Wyle Y.W. Tsan; "Optimal synthesis of saturating sampled-data systems using integral performance index", *International Journal of Control.*, vol. 14, No. 1, pp. 33-42, 1971.
7) Yeh, H.H.; "Optimal control with partially specified input functions", *International Journal of Control*, vol. 16, No. 1, pp. 71-80, 1972.
8) Yeh, H.H. and R.J. Kuhler; "Additional properties of an extended maximum principle", *International Journal of Control*, vol. 17, No. 6, pp. 1281-1286, 1973.
9) Whitbeck, R.F.; "Analysis of digital flight control system with flying qualities applications", AFFDL-TR-78-115, 1978, Flight Dynamics Laboratory, Wright-Patterson Air Force Base, Ohio.