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Inventory Model with Partial Backorders*

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Abstract

This article presents a deterministic inventory model for situations in which, during the stockout period, a fraction β of the demand is backordered and the remaining fraction $1-\beta$ is lost. By defining a time proportional backorder cost and a fixed penalty cost per unit lost, a convex objective function representing the average annual cost of operating the inventory system is obtained. The optimal operating policy variables are calculated directly. At the extremes $\beta = 1$ and $\beta = 0$ the model presented reduces to the usual backorders and lost sales case, respectively.

List of Symbols

- d Demand per year.
- R Total demand per cycle.
- S Total demand per cycle during the stockout period.
- Q Order quantity.
- h Inventory carrying cost per item per year.
- A Fixed ordering cost per inventory cycle.
- p Penalty cost of a lost sale including the lost profit, $p > 0$.
- π Shortage cost per unit time per backorder, $\pi > 0$.

1. Introduction

The problem of determining economic order quantity in inventory systems has been treated extensively in the literature. But most of the results are on two general situations regarding the

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demand process when the system is out of stock : either, all demand during the stockout period is backordered (backorder case); or, all demand during the stockout period is lost (lost sales case).

However, in many real inventory systems, demand can be partially captive. In this case only a portion of unsatisfied demand will be satisfied from the next shipment. If demand is completely captive, all demands not satisfied are filled upon the receipt of replenishment stock. If demand is not captive, unsatisfied demand may be completely lost. Thus it may be reasonable to assume that only a fraction $\beta (0 \leqslant \beta \leqslant 1)$ of the demand during the stockout period can be backordered, and the remaining fraction $1-\beta$ is lost. Furthermore, a numerical example shows that making the usual assumptions of all backorders (or lost sales) when in fact a mixture of the two exists can significantly affect inventory costs.

Inventory models which consider a mixture of backorders and lost sales have been suggested (without solving) by several authors (Fabrycky and Banks 1967, Jelen 1970). Montgomery *et al.* (1973) suggested a solution procedure using 'non-singular transformation' of the cost function and complicated two stage minimization. In the first stage, they minimize the cost along the 'ray' $V=bU$. Once the minimum along each ray of this form is found, they choose the best ray. Similarly, Rosenberg (1979) formulated the problem in terms of 'fictitious' demand rate'. An optimal solution is sought using 'decomposition by projection' in the stepwise manner specified.

In this paper, the problem is reformulated by defining a time proportional backorder cost and a fixed penalty cost per unit lost; and solved *directly*. This simplification results from the different choice of control variables and noting the convexity of the cost model.

2. Model

We consider the single echelon, single item, constant demand case.

We assume that the fraction of the demand backordered during the stockout period is known

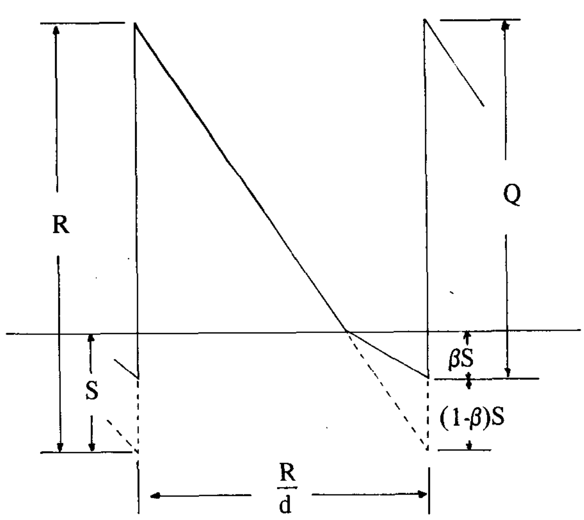

Figure 1. Behaviour of the inventory system.

and constant. The total number of demands backordered per inventory cycle is βS, and the total number of demands lost is $(1-\beta)S$. The behaviour of the system is geometrically represented in Fig. 1.

The annual revenue received will depend on the length of time for which the system is out of stock, and hence on the operating doctrine. Thus, one would not necessarily obtain the same operating policy from a model which minimizes average annual cost as from a model which maximizes the average annual profit. However, by defining lost sales penalty costs to include the lost profits, minimum cost and maximum profit formulations yield identical results (Hadley and Whitin 1963).

The average annual cost which is the sum of ordering, carrying, and stockout costs including lost profit is

$$K(R,S,\beta) = [2dA + h(R-S)^2 + 2(1-\beta)pdS + \beta\pi S^2]/2R\ ; \quad R \geqslant S \geqslant 0 \tag{1}$$

K is convex (see Appendix). A sufficient condition that R and S be optimal is that they satisfy

$$\partial K/\partial R = -[2dA + h(R-S)^2 + 2(1-\beta)pdS + \beta\pi S^2]\ /2R^2 + h(R-S)/R = 0$$

or

$$R^2 = [2dA + 2(1-\beta)pdS + \beta\pi S^2]/h + S^2 \tag{2}$$

and

$$\partial K/\partial S = [-h(R-S) + (1-\beta)pd + \beta\pi S]/R = 0$$

or

$$R = [(1-\beta)pd + (1 + \beta\pi)S]/h \tag{3}$$

Solving eqns. (2) and (3) simultaneously to eliminate R, we obtain

$$(h + \beta\pi)S^2 + 2(1-\beta)pdS + \{[(1-\beta)pd]^2 - 2hdA\}/\beta\pi = 0$$

Solution of this quadratic equation yields the result

$$S^* = [-(1-\beta)pd + \sqrt{(h\{2dA(h + \beta\pi)} - [(1-\beta)pd]^2\}/\beta\pi)]\,(h + \beta\pi) \tag{4}$$

Since $\partial^2 K/\partial S^2 > 0$, in the event that the S value computed from eqn. (4) is not positive (or real), then the S^* lies on the boundary and $S^* = 0$. This occurs when the quantity inside the square root in eqn. (4) is less than $[(1-\beta)pd]^2$ or

$$0 \leqslant \beta \leqslant 1 - (1/p)\sqrt{(2hA/d)}^{(\dagger)} \tag{5}$$

In this case, eqn. (1) is reduced to the basic EOQ model and

(†) Note that when $1-(1/p)\sqrt{(2hA/d)} < \beta = 0$, there should not be any inventory system since this condition implies that incurring the cost of lost sales all the time (dp) is cheaper than operating a system where lost sales are never incurred ($\sqrt{2hdA}$) (Hadley and Whitin 1963).

$$Q^* = R^* = \sqrt{(2dA/h)} \qquad (6)$$

If the S^* computed from eqn. (4) is positive, the optimal value of R can be determined explicitly by using (3) to eliminate S in (2). This yields $R^2 = \{ 2dA(h+\beta\pi) - [(1-\beta)pd]^2 \} / h\beta\pi$. Thus the optimal value of R is

$$R^* = \sqrt{(2dA(h + \beta\pi) - [(1-\beta)pd]^2)}/\sqrt{(h\beta\pi)} \qquad (7)$$

($R^* \geqslant S^*$ from eqns. (2) and (3)) and the optimal order quantity is

$$Q^* = R^* - (1-\beta)S^* \qquad (8)$$

It is noted from eqn. (7) that when $\beta \geqslant 1-(1/p)\sqrt{(2A/d)}\sqrt{(h+\beta\pi)}$, there exists a real R^* satisfying (7). However, since the right hand side of this inequality is smaller than the right hand side of eqn. (5), real R^* exists whenever $S^* \geqslant 0$.

3. Numerical Example and Sensitivity Analysis

It is very easy to demonstrate that inventory systems are sensitive to assumptions regarding the nature of demand during the stockout period. For example, suppose that an item has the following characteristics

d = 200 units/year
h = \$3/year
A = \$50/order
p = \$2/unit lost
π = \$1/year per backorder

When $\beta = 0$, i.e., unsatisfied demands are all lost, eqn. (5) applies and the problem reduces to the basic EOQ situation. Thus $Q^* = R^* = 82$ units from eqn. (6). This result coincides with that of Hadley and Whitin (1963). Up to $\beta = 0.3876$, eqn. (5) still applies.

When $\beta = 0.5$, half of the unsatisfied demand will be satisfied from the next shipment. From eqns. (4), (7), and (8), $S^* = 64, R^* = 141, Q^* = 109$. From eqn. (1), $K(R = 141, S = 64, \beta = 0.5)$ = \$232. We see that it is cheaper to order 109 units every cycle and set the reorder point such that replenishment stock arrives when the cumulative shortage reaches 64 units.

Table 1 shows the optimal operating policies associated with different values of β, fraction backordered. As the β increases, S^* and Q^* increase but the minimum annual average cost decreases.

β	0	0.3876	0.5	0.9	1
S^*	0	0	64	119	122
R^*	82	82	141	168	163
Q^*	82	82	109	156	163
$K(R^*, S^*, \beta)$	\$245	\$245	\$232	\$147	\$122

Table 1. Sensitivity of β (fraction backordered).

Frequently, inventory managers assume that $\beta = 1$ even though it does not. That is, they assume that all demands during the stockout period can be backordered even though this is not strictly true. To carry this example further, suppose we do this for the above data when the β is actually 0.5. In this case, the managers would assume erroneously $Q_1{}^* = R_1{}^* = 163, S_1{}^* = 122$. The true average annual cost that would result from using $R_1{}^*$ and $S_1{}^*$ may be found by substituting $R_1{}^*$ and $S_1{}^*$ in eqn. (1) as $K(R = 163, S = 122, \beta = 0.5) = \$249$. Thus, failure to use the appropriate model has cost management \$249–\$232 = \$17 per year for this single item. The effects of this in a multi-item inventory may be quite substantial.

Figure 2 shows the effect of erroneous assumption of β on the average annual cost of operating the inventory system.

4. Conclusion

This article has treated inventory process in which a fraction β of the demand during the stockout period is backordered and the remainder is lost. It is felt that the model represents the real nature of inventory systems and is useful for the practical solution of inventory problems. Erroneous assumption of β, the fraction backordered, appears to be very sensitive to the annual average cost of operating the inventory system.

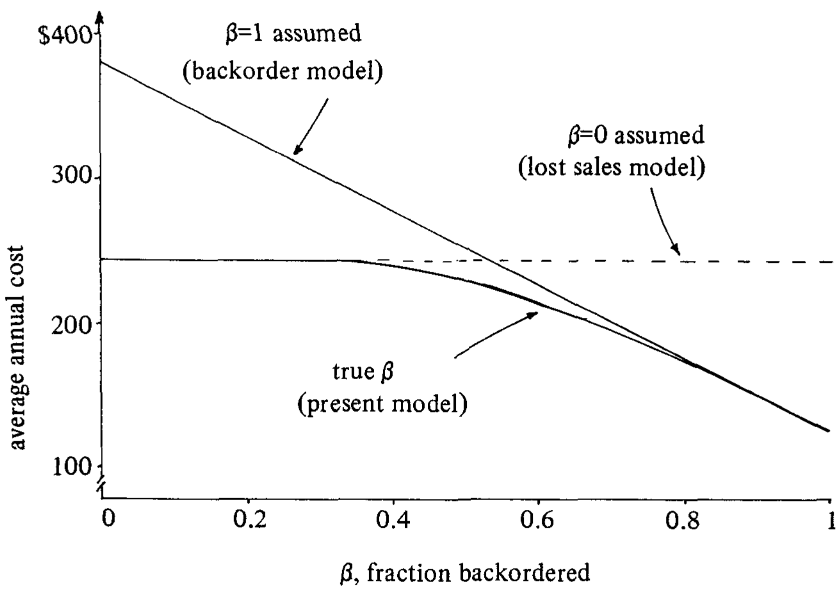

Figure 2. Effect of erroneous assumption of β on annual cost.

Appendix

Proof of the convexity of K

K is continuously twice differentiable. When $\beta \leqslant 1-(1/p)\sqrt{(2hA/d)}$, eqn. (5) holds, and the problem degenerates to the basic EOQ model: $K(R) = (2dA + hR^2)/2R$. In this case, the convexity of K is obvious. When $\beta > 1-(1/p)\sqrt{(2hA/d)}$ or $[(1-\beta)pd]^2 < 2hdA$

$$\partial^2 K/\partial R^2 = [2dA + hS^2 + 2(1-\beta)pdS + \beta\pi S^2]/R3$$

$$\partial^2 K/\partial S^2 = (h + \beta\pi)/R$$

$$\partial^2 K/\partial R\,\partial S = -[(h + \beta\pi)S + (1-\beta)pd]/R^2$$

The determinant of Ks hessian matrix, $|H|$, is

$$|H| = \partial^2 K/\partial R^2 \, \partial^2 K/\partial S^2 - (\partial^2 K/\partial R \partial S)^2$$

$$= \{2hdA + 2dA\beta\pi - [(1-\beta)pd]^2\}/R^4$$

$$\geqslant 2dA\beta\pi/R^4 \geqslant 0$$

Therefore, Ks hessian is positive semi-definite and K is convex.

References

1. Fabrycky, W.J., and Banks, J., 1967, Procurement and Inventory Systems (New York: Reinhold Publishing Corporation).
2. Hadley, G., and Whitin, T.M., 1963, Analysis of Inventory Systems (New Jersey: Prentice-Hall, Inc.).
3. Jelen, F.C. (editor), 1970, Cost and Optimization Engineering (New York: MacGraw-Hill, Inc.).
4. Montgomery, D.C., Bazaraa, M.S., and Keswani, A.K., 1973, Naval Res. Logist. Q., 20(2).
5. Rosenberg, D., 1979, Naval Rės. Logist. Q., 26(2).