Optimal Capacity Expansion and Operation With Alternative Financing

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Abstract

This paper is concerned with the optimal control of dynamic expansion and operation of a single capacity under deterministic demand. Three cases of financing mode are considered : unlimited borrowing, debt aversion, and self financing. Using the net revenue as the objective function, the optimal paths of production and investment are analytically derived.

1. Introduction

The growth of a firm is dynamic ; it consumes resources, generates greater amount of product after certain lags, and involves investment decisions for the possible expansions in the future. Hence an important class of the dynamic theory of the firm is the problem of investment and production.

A seminal work on this category is due to Arrow, Beckman, and Karlin [1] who formulated the capacity expansion problem in continuous time and obtained optimal capacity expansion policies. Thompson and George [8] formulated a dynamic model of the firm encompassing operations and investments, and solved the problem by using the Pontryagin's maximum principle. Perrakis and Sahin [7] derived the optimal capacity expansion path for a monopoly firm in an irreversible investment situation. They assumed that the marginal returns to capacity are always positive and thereby ruled out the possibility of firms holding idle capacity, so any additional revenue generated by its use means additional profits.

Capacity expansion with the revenue function which includes idle capacity cost is considered in this paper. The optimal policies of investment and operation are derived under various financial cases of unlimited borrowing, debt aversion, and self financing by using the Pontryagin's maximum principle.

2. Model with Unlimited Borrowing

2.1. Nomenclature

The following notations are used throughout this paper.

A. Exogenous parameters

c = unit capacity purchase price

i = interest rate

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r = depreciation rate
M = upper bound on purchase rate of new capacity
b = dividend payout ratio, $0 \leq b \leq 1$
w = wage rate
m = idle capacity cost rate

B. Control variables

u (t) = scale of operation, $0 \leq u(t) \leq 1$
v (t) = capacity purchase rate or investment rate
L (t) = labor input

C. State variables

X (t) = capacity of firm
D (t) = net debt
R [u (t) X (t), L (t)] = revenue function

The revenue function is assumed to be concave differentiable with continuous first and second partial derivatives on an open set contained in the nonnegative orthant. It will be also assumed that $R_X|_{0,L} > -m$ and $R_L|_{uX,0} > w$ for uX in some open neighborhood of 0.

2.2. Formulation

The objective of the firm is to maximize the discounted value of net revenue, i. e., revenues from sales less the cost of production, investment, interest payments, and cost of idle capacity. The discounted value of capacity at the end of the finite planning horizon is added into the objective function to ensure the continuity of production after the planning horizon. The model is constrained by two differential equations describing the changes in capacity and debt, and there are inequality constraints on controls and state variables.

Then the capacity expansion model for a firm with unlimited borrowing financial situation can be formulated as follows :

$$\text{Maximize } J = \int_0^{t_1} \{R[u(t)X(t), L(t)] - wL(t) - iD(t) - cv(t) - mX(t)(1-u(t))\} e^{-it}\, dt + cX(t_1)e^{-it_1}$$

subject to

$$\dot{X}(t) = v(t) - rX(t)$$
$$\dot{D}(t) = iD(t) + wL(t) + cv(t) - R[u(t)X(t), L(t)] - mX(t)(1-u(t))$$
$$0 \leq u(t) \leq 1$$
$$0 \leq v(t) \leq M$$

where M is an arbitrary constant and the initial conditions $X(0) > 0$, $D(0)$ are assumed to be given.

To apply the Pontryagin's maximum principle, the Hamiltonian can be defined by :

$$H = \lambda_0 [R - wL - iD - cv - mX(1-u)] e^{-it} + P_1(v - rX) + P_2(iD + wL + cv + mX(1-u) - R) + \lambda_1 u + \lambda_2(1-u) + \lambda_3 v + \lambda_4(M-v)$$
$$G = -\lambda_0 cX(t_1)e^{-it_1}$$

where λ_i and P_i are Lagrange multipliers and adjoint variables, respectively, and $\lambda_0 = 1$.

Then the necessary and transversality conditions become

$$H_u = X(R_x + m)(e^{-it} - P_2) + \lambda_1 - \lambda_2 = 0 \quad (1)$$
$$H_L = (R_L - w)(e^{-it} - P_2) = 0 \quad (2)$$
$$H_v = c(P_2 - e^{-it}) + P_1 + \lambda_3 - \lambda_4 = 0 \quad (3)$$
$$P_1(t_1) = ce^{-it_1}$$
$$P_2(t_2) = 0 \quad (4)$$
$$\dot{P}_1 = P_1 + \{m - (m + R_x)u\}(e^{-it} - P_2) \quad (5)$$
$$\dot{P}_2 = i(e^{-it} - P_2) \quad (6)$$
$$\lambda_1 u = \lambda_2(1-u) = \lambda_3 v = \lambda_4(M-v) = 0,$$
$$\lambda_i \geq 0 \text{ for } i = 1, \cdots\cdots 4 \quad (7)$$

and v maximizes H at every point within the space defined by the two linear constraints. Since both of the multipliers λ_3 and λ_4 cannot be positive, it can be easily shown from the equation (3) that

$$v = 0 \quad \text{if } P_1 \leq c(e^{-it} - P_2) \quad (8)$$
$$v = M \quad \text{if } P_1 \geq c(e^{-it} - P_2) \quad (9)$$
$$0 \leq v \leq M \quad \text{if } P_1 = c(e^{-it} - P_2) \quad (10)$$

It can also be shown that the multiplier P_2 is negative, i. e.,

$$e^{-it} - P_2 > 0 \quad (11)$$

since the integration of the appropriate dynamic equation with $P_2(t_1) = 0$ yields $P_2 = -ie^{-it}(t_1 - t)$. The positivity of $e^{-it} - P_2$ implies in both cases that along the optimal path, $R_L = w$.

Similarly, since both of λ_1 and λ_2 cannot be positive,

$u=1$ if $R_x(uX, L) > -m$ and $\lambda_2 > 0$ (12)

$0 < u \leq 1$ if $R_x(uX, L) = -m$ and $\lambda_1 = \lambda_2 = 0$ (13)

and along the optimal path, uX is always positive.

The equation $R_L = w$ contains the variables uX and L only. If there is a unique solution $L = L(uX)$, then the control variable L is completely determined as a function of u and X.

Proposition 1. Let $L(uX)$ be the (unique) solution of the equation $R_L = w > 0$. Then the function $R - wL + muX$ of the variable uX is positive, concave and differentiable on some non-empty open interval $(0, \varepsilon)$.

Proof. The concavity of R and the condition $R_L(uX, 0) > w$ guarantees the positivity of $L(uX)$ for uX in some open neighborhood of 0.

Since the second partial derivative of R with respect to L satisfies that $R_{LL}(uX, L) < 0$ for all uX and L on the nonnegative orthant, it follows that for uX in some open interval $(0, \varepsilon_1)$, $R_{LL}(uX, L(uX)) < 0$.

Since R_{LL} and R_{LX} exist and are continuous, it follows that $L(uX)$ is continous and $L_x = dL(uX)/d(uX)$ exists and continuous within the interval.

To prove concavity it is sufficient to show that for any uX and $\overline{uX}$ on $(0, \varepsilon_1)$.

$$R(uX, L(uX)) - wL(uX) + muX - R(\overline{uX}, (ux)) + wL(\overline{uX}) - m\overline{uX} \leq [d/d(\overline{uX})\{R(\overline{uX}, L(\overline{uX})) - wL(\overline{uX}) + m\overline{uX}\}][uX - \overline{uX}] = \{R_x + R_L L_x - wL_x + m\}[uX - \overline{uX}] = (R_x + m)(uX - \overline{uX})$$

Then the inequality becomes

$$R(uX, L(uX)) - R(\overline{uX}, L(\overline{uX})) \leq (R_x + m)(uX - \overline{uX}) + w(L(uX) - L(\overline{uX})) - m(uX - \overline{uX})$$

Finally, define $N(uX) = R(uX, L(uX)) - wL(uX) + muX$, then $N_x = R_x + m > 0$ at $uX = 0$, where N_x and R_x denote to the partial derivatives with respect to uX. Hence, $N(uX)$ is positive on some nonempty subinterval $(0, \varepsilon)$ of $(0, \varepsilon_1)$. Q.E.D.

The necessary conditions can now be reformulated in terms of N by substituting N, N_x and 0 instead of $R - wL + muX$, $R_x + m$, and $R_L - w$.

2.3. Optimal Path

From the equations (12) and (13), and the concavity of N, it follows that the optimal operating or production policies, u^* are

$u^* = 1$ if $N_x > 0$ or $0 < u^* \leq 1$ if $N_x = 0$.

If $N(uX)$ is strictly concave, then the equation $N_x = 0$ gives an unique point $uX = \hat{X}$. Set $\hat{X} = \inf\{uX, N_x = 0\}$ then the optimal polices can be given by

$u^* = \hat{X}/X$ for $X > \hat{X}$ and $u^* = 1$ for $X \leq \hat{X}$.

In order to determine the optimal investment policies, the following cases can be considered.

A. Case of $X(0) < \hat{X}$

For all the paths with the initial capacity smaller than $\hat{X}$

$$P_1 = e^{rt}\{C - \int_0^{t_1}[N_x - m](1 + i(t_1 - T))e^{-(i+r)T}dT\}$$

from the equations (4) and (5), and if $X(t_1) \leq \hat{X}$,

$$C = ce^{-(i+r)t_1} + \int_0^{t_1}(N_x - m)(1 + i(t_1 - T))e^{-(i+r)T}dT$$

$$P_1 = e^{rt}[ce^{-(i+r)t_1} + \int_t^{t_1}(N_x - m)(1 + i(t_1 - T))e^{-(i+r)T}dT.$$

Cases (8), (9) and (10) hold respectively if

$P_1 \lesseqgtr ce^{-it}(1 + i(t_1 - t))$

or

$$\int_t^{t_1}[N_x - m - c(i+r) + ci/(1 + i(t_1 - T))](1 + i(t_1 - T))e^{-(i+r)T}dT \lesseqgtr 0 \qquad (14)$$

For $t \in [0, t_1]$

$$c(i+r) + ci/(1 + i(t_1 - t)) \in [c(i+r) + ci/(1 + it_1), 2ci + cr] \qquad (15)$$

where for large t_1, the interval becomes $[c(i+r), 2ci + cr]$.

Then, the optimal investment policy is

$v^* = 0$ for all t if $N_x|_{x=0} < c(i+r) + ci/(1 + it_1) + m$

$v^* = M$ for all t if $\lim_{X \to \infty} N_x > 2ci + cr + m$

To find the paths $X(t)$, it is necessary to restrict the form of N. Suppose

$$N_X|_{X=0} > 2ci + cr + m \quad (16)$$

and

$$\lim_{X\to\infty} N_X < c(i+r) + ci/(1+it_1) + m \quad (17)$$

then it is possible to find two points $X_1(0)$ and $X_1(t_1)$, such that $0 < X_1(t_1) < X_1(0) < \hat{X}$ with

$$N_X|_{X=X_1(t_1)} = 2ci + cr + m$$

$$N_X|_{X=X_1(0)} = c(i+r) + ci/(1+it_1) + m$$

Suppose, in addition, that the depreciation rate satisfies

$$r > -\dot{X}/X = -ci^2/(1+i(t_1-t))^2 N_{XX}$$

for all $X \varepsilon [X_1(t_1), X_1(0)]$, then it is possible to find a capacity $X_1(t)$ satisfying the equation

$$N_X = c(i+r) + ci/(1+i(t_1-t)) + m \text{ for all } t \varepsilon [0, t_1] \text{ and } r > -\dot{X}_1/X_1.$$

With these restrictions, the optimal investment policy can be given as follows :

a) If $X(0) < X_1(0)$, then $v^* = M$ for the time interval $[0, t']$. If $M/r + (X(0) - M/r)e^{-rt_1} < X_1(t_1)$ then $t' = t_1$: otherwise t' is determined by the equation $M/r + (X(0) - M/r)e^{-rt'} = X_1(t')$. For $t \varepsilon [t', t_1]$, $v = \dot{X}_1 + rX_1$ which implies that for this interval the equality sign will hold in (14).

b) If $X_1(0) < X(0) \leqslant \hat{X}$ for $t \varepsilon [0, t_j]$, then $v^* = 0$ and t_j is determined from $X(0) e^{-rt_j} = X_1(t_j)$; if no such t_j exists then $t_j = t_1$. For $t \varepsilon (t_j, t_1]$, $v^* = \dot{X}_1 + rX_1$.

c) If $X(0) = X_1(0)$, $v^* = \dot{X}_1 + rX_1$ for $t \varepsilon [0, t_1]$.

B. Case of $X(0) > \hat{X}$

If the initial capacity is greater than $\hat{X}$, i.e., $X(0) > \hat{X}$, then (13) and (5) imply

$$P_1 = e^{rt}(C + m\int_0^t (1+i(t_1-T))e^{-(i+r)T} dT)$$

for all the path with $X > \hat{X}$.

If $X(t_1) > \hat{X}$,

$$C = ce^{-(i+r)t_1} - m\int_0^{t_1} (1+i(t_1-T))e^{-(i+r)T} dT.$$

So the following inequality is satisfied

$$\int_t^{t_1} [-m - c(i+r) - ci/\{1+i(t_1-T)\}][1+i(t_1-T)]e^{-(i+r)T} dT < 0.$$

Hence, the optimal policy is $v^* = 0$.

Proposition 2. If $X(0) > \hat{X}$ and $X(t_1) < \hat{X}$, (8) still holds for $t \varepsilon [0, t')$ where t' satisfies $X(0)e^{-rt'} = \hat{X}$.

Proof. It follows that

$$C = P_1(t')e^{-rt'} - m\int_0^{t'} (1+i(t_1-T))e^{-(i+r)T} dT$$

$$P_1(t') \leq ce^{-it'}(1+i(t_1-t'))$$

$$C \leq ce^{-(i+r)t'}(1+i(t_1-t')) - m\int_0^{t'} (1+i(t_1-T))e^{-(i+r)T} dT$$

$$\leq ce^{-(i+r)t}(1+i(t_1-t)) - m\int_0^t (1+i(t_1-T))e^{-(i+r)T} dT$$

Hence,

$$P_1(t) = e^{rt}[P_1(t')e^{-rt'} - m\int_0^{t'}(1+i(t_1-T))e^{-(i+r)T} dT + m\int_0^t (1+i(t_1-T))e^{-(i+r)T} dT]$$

$$\leq e^{rt}[ce^{-(i+r)T}(1+i(t_1-t)) - m\int_0^t (1+i(t_1-T))e^{-(i+r)T} dT + m\int_0^t (1+i(t_1-T))e^{-(i+r)T} dT]$$

$$= ce^{-it}(1+i(t_1-t)). \quad \text{Q. E. D.}$$

If $X(t_1) < \hat{X}$, then $v^* = 0$ for some $t' \varepsilon (0, t_1)$ with t' satisfying $X(0)e^{-rt'} = \hat{X}$, and for $t \varepsilon [t', t_j]$ where t_j is determined from $\hat{X}e^{-r(t_j - t')} = X_1(t_j)$. If there doesn't exist such t_j then $t_j = t_1$. Also for $t \varepsilon (t_j, t_1)$, $v^* = \dot{X}_1 + rX_1$

2.4. Economic Interpretation

For the unlimited borrowing financial case, the firm should invest maximum as much as possible whenever the marginal revenue of capacity R_X is greater than the economic rent, $c(i+r) + ci/[1+i(t_1-t)]$, where the second term represents the borrowing rent. The firm should not invest whenever the opposite is the case. If the marginal revenue is the same as the economic rent at time t, then the firm will invest singular amout to maintain this equality from time t to the end of the planning horizon.

For the optimum production policies, the firm should operate at full capacity whenever the marginal revenue of capacity plus excess capacity cost is positive. If it is equal to zero, the production level is determined by the ratio of $\hat{X}$ to X.

It turns out that the direction of the optimal policies is a function of the capacity. It should be noted that the assumptions of revenue function must be relaxed if idle capacity cost is allowed.

3. Model with Debt Aversion

If a firm does not borrow at all but lend to others, D (t) can not be positive and the maximum investment should be the total money available to the firm without any debt. Thererore, the constraints which should be added to the previous unlimited borrowing case are as follows :

$0 \leq v \leq R - wL - iD - mX(1-u)/c$

$\dot{D} \leq 0$

$D \leq 0$

$X(0) \geq 0$

$D(0) \leq 0$

The corresponding Hamiltonians are

$H = [R - wL - iD - cv - mX(1-u)]e^{-it} + P_1(v - rX) + P_2(iD + wL + cv + mX(1-u) - R) + \lambda_1 u + \lambda_2(1-u) + \lambda_3 v + \lambda_4((R - wL - iD - mX(1-u))/c - v)$

$G = -cX(t_1)e^{it_1}$

Then the necessary and transversality cons are

$H_u = X(R_X + m)(e^{-it} - P_2 - \lambda_4) + \lambda_1 - \lambda_2 = 0$ (19)

$H_L = (R_L - w)(e^{-it} - P_2 + \lambda_4) = 0$ (20)

$P_1(t_1) = ce^{it_1} \quad P_2(t_1) = 0$ (21)

$\dot{P}_1 = P_1 r + (m - (m + R_X)u)(e^{-it} - P_2 + \lambda_4)$ (22)

$\dot{P}_2 = -iP_2 + ie^{-it} + i\lambda_4$ (23)

$\lambda_1 u = \lambda_2(1-u) = \lambda_3 v = \lambda_4(R - wL - cv - mX(1-u)) = 0, \; \lambda_i \geq 0 \; i = 1, \cdots, 4$ (24)

and the optimal investment v^* satisfies

$(v^* - v)[P_1 + \lambda_3 - c(e^{-it} - P_2 + \lambda_4)] > 0$ (25)

for all v's satisfying $(R - wL - iD - mX(1-u))/c \geq v \geq 0$.

It can be easily shown that the multipliers P_2 is negative, i. e., $e^{-it} - P_2 > 0$ and

$P_2 = -ie^{-it}(t_1 - t) - ie^{-it}\int_t^{t_1} \lambda_4 e^{-iT} dT$ (26)

From $\lambda_4 \geq 0$ and the positivity of $e^{-it} - P_2$, $R_L = w$ along the optimal path.

Thus the optimal investment policies are

$v^* = 0$ if $P_1 + \lambda_3 \leq c(e^{-it} - P_2)$ (27)

$0 \leq v^* \leq (N - iD - mX)/c$ if $P_1 + \lambda_3 = c(e^{-it} - P_2 + \lambda_4)$ (28)

$v^* = (N - iD - mX)/c$ if $P_1 \geq c(e^{-it} - P_2 + \lambda_4)$ (29)

Under the conditions (16), (17), (18) and the strict concavity of N (X) for $X \in [X_1(t_1), X_1(0)]$, the optimal investment policies are the same as the unlimited-borrowing case for $X(0) \geq X_1(0)$. For $X(0) < X_1(0)$, the optimum policies can be derived as.

$v^* = \text{maximum}$ if $X(t) < X_1(t)$

$v^* = 0$ if $X(t) > X_1(t)$

$v^* = \dot{X}_1 + rX_1$ if $X(t) = X_1(t)$

4. Model with Self Financing

A particular case of debt aversion model is self financing situation, i. e., a firm does not borrow and it can invest only the net revenue which is the gross revenue minus dividend. Then the model can be formulated as follows :

Maximize $J = \int_0^{t_1} (R(uX, L) - wL - cv - mX(1-u))e^{-it}dt + cX(t_1)e^{it_1}$

subject to

$\dot{X} = v - rX$

$0 \leq v \leq (b/c)[R(uX, L) - wL - mX(1-u)]$

$0 \leq u \leq 1$

Then the Hamiltonian, and the necessary and transversality conditions are

$H = (R - wL - cv - mX(1-u))e^{-it} + P_1(v - rX) + \lambda_1 u + \lambda_2(1-u) + \lambda_3 v + \lambda_4[b/c(R - wL - mX(1-u)) - v]$

$H_u = X(R_X + m)(e^{-it} + \lambda_4 b/c) + \lambda_1 - \lambda_2 = 0$ (30)

$H_L = (R_L - w)(e^{-it} + b/c\,\lambda_4) = 0$ (31)

$H_v = P_1 - ce^{-it} + \lambda_3 - \lambda_4 = 0$ (32)

$P_1(t_1) = ce^{-it_1}$ (33)

$\dot{P}_1 = P_1 r + [m - (m + R_X)u](e^{-rt} + b/c\,\lambda_4) = 0$ (34)

$\lambda_1 u = \lambda_2(1-u) = \lambda_3 v = \lambda_4(b/c[R(uX, L) - wL$

$-mX(1-u)]-v)=0$ (35)

$\lambda_i > 0,\ i=1,\cdots\cdots,\ 4$

Using the previous result that $R_L=w$ and uX is positive along the optimal path, the problem can be simplified by substituting the optimal value $L(uX)$ instead of L and solving the problem with respect to two control variables.

Similar to the previous models, the optimum investment policies can be derived as follows :

A. When $X(0) < \hat{X}$

$v^*=0$ if $N_X|_{X=0} < m+c(i+r)$

$v^*=M$ if $\lim_{X\to\infty} N_X > m+c(i+r)$

For unsaturated solution, it is also necessary to restrict the form of N.

Suppose,

$N_X|_{X=0} > m+c(i+r)$ (36)

and

$\lim_{X\to\infty} N_X < m+c(i+r)$ (37)

then it is possible to find the point X_2 from $N_X|_{X=X_2(t)}=c(i+r)+m$. Suppose, in addition, that N is concave and $r>0$ for $X=X_2$; then it is possible to find a capacity that is a continuous function $X_2(t)=X_2$ satisfying the equation $N_X=m+c(i+r)$ for all $t\varepsilon[0, t_1]$ and $r\geqq-\dot{X}_2/X_2=0$. With these restrictions the optimal investment policy is :

a) When $X(0)=X_0<X_2$. The policy is $v^*=M$ for the time interval $[0, t'']$. If $M/r+(X_0-M/r)e^{-rt_1}\leqq X_2$, then $t''=t_1$ otherwise t'' is determined by the equation $M/r+(X_0-M/r)e^{-rt''}=X_2$. $v^*=rX_2$ for $t\varepsilon(t'', t_1]$.

b) When $X_2<X(0)\leqq X$. For $t\varepsilon[0, t'']$, $v^*=0$ and t'' is determined from $X(0)e^{-rt''}=X_2$: If no such t'' exists then $t''=t_1$. For $t\varepsilon(t'', t_1]$, $v^*=rX_2$.

c) When $X(0)=X_2$. $v^*=rX_2$ for all $t\varepsilon[0, t_1]$

B. When $X(0) > \hat{X}$

$v^*=0$ if $X(t_1) > \hat{X}$

If $X(t_1) < \hat{X}$ then for some $t''\varepsilon(0, t_1)$, $X(0)e^{-rt''}=\hat{X}$ and $Xe^{-rt_2}=X_2$ from which t_2 is determined. If no such t_2 exists then $t_2=t_1$ and $v^*=0$ for $t''\varepsilon(0, t_1]$. If such t_2 exists, then for $t\varepsilon(t_2, t_1]$, $v^*=rX_2$.

Choosing $b(N-mX)/c$ instead of M, similar results of previous models can be obtained for each cases of $X(0)>X_2$ and $X(0)<X_2$.

Thus the optimal policy turns out to be relatively simple in both cases.

$v^*=$ maximum if $X(t)<X_2$

$v^*=0$ if $X(t)>X_2$

$v^*=rX_2$ if $X(t)=X_2$

5. Conclusion

For the simple but diametrically opposite financial cases considered in this paper, it turns out that the direction of the optimal policies of investment and operation is a function of the capacity even though the optimal paths are different in general.

The optimal expansion path has the following properties ; the firm should invest maximum in new capacity whenever the marginal revenue of capacity is greater than the economic rent, and it will not invest whenever the opposite is the case. If the marginal revenue of capacity is the same as the economic rent, then singular purchase rate is maintained.

The optimum paths are crucially influenced by the financing mode and idle capacity. The financing mode determines the maximum available capital that the firm can expend while the idle capacity alters the marginal revenue of capacity.

Models with general financing conditions, oligopolistic competition, and various regulations can be the areas for further study.

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