A Sufficient Condition for the Independence of Non-Stationary Demand Process and Inventory Position Process under 〈Q,r〉 Systems

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Abstract

Under a continuous-review inventory system 〈Q, r〉, the inventory position process was proved to be asymptotically independent of the general renewal demand processes, when the two processes form an asymptotic unimodel joint distribution.

The analytical technique implemented through this work seems to be more like general, and so the periodic-review system 〈R, r, T〉 can be similarly investigated.

In conclusion, the results may be evaluated to direct to the analytical analyses of some inventory systems which have been treated under some restrictions on demand processes.

1. Introduction

Inventory systems are operated largely based on some operating policies concerning review systems and ordering rules. The so-called transactions-reporting (continuous-review) systems and periodic-review systems are commonly used for inventory system review.

In both inventory systems the inventory position $\{IP_t : t > 0\}$ totally depends upon the demand process $\{N_t : t > 0\}$. Furthermore, the so-called net inventory process $\{NIS_t : t > 0\}$ can be defined as $NIS_t = IP_{t-u} - D(t-u, t]$ for a lead time $u > 0$, where $D_{(t-u,t]} \equiv \tilde{N}_t - N_{t-u}$. Therefore, once it is verified that $\{IP_{t-u}\}$ and $\{D_{(t-u,t]}\}$ are mutually independent of each other, the analysis of $\{NIS_t\}$ will become straightforward, from which the cost process can be immediately derived whose average one may seek.

In such research direction, the inventory position process associated with stationary Poisson demand process has been specified in Hadley and Whitin [1]. Sahin [3] has extended the study for continuous-review 〈s, S〉 inventory systems with general renewal inter-arrival and demand processes and a constant lead time. However, continuous-review 〈Q, r〉 inventory systems are not treated in literature.

It will be proved that for a 〈Q, r〉 inventory system with general renewal interarrival and demand processes $\{N_t : t > 0\}$ and a constant lead time $u > 0$, $\{IP_{t-u}\}$ and $\{D_{(t-u,t]}\}$ are asymptotically independent of each other.

2. System Analysis

When demands arrive at time points t_1, t_2

……, $(0<t_1<t_2)$, the sucessive inter-arrival times $\{X_i : i>1\}$ are defined as $X_1=t_1$, $X_2=t_2-t_1$, ……, $X_n=t_n-t_{n-1}$,…… Let N_t be cumulative demand by time $t\geq 0$. Then $\{N_t : t\geq 0\}$ is a discrete-valued continuous-parameter counting process with sample paths non-decreasing in unit steps.

An inventory position IP_t at time t totally depends upon the demand process $\{D_t : t\in T\}$, where the parameter set T is the index set of the process. If the inventory system $\langle Q, r\rangle$ is started with $IP_0=r+i\,(i=1,2,\cdots\cdots,Q)$ at time $t=0$, then $IP_{t-u}=r+j\,(j=1,2,\cdots\cdots,Q)$ at time $t-u>0$ can be reached after the $(i-j)^{+}$ or $\{i+(m-1)Q+(Q-j)\ ;\ m=1,2,\cdots\cdots\}$ demand materialization by time $t-u$, where m denotes the total number of order replacements by time $t-u$ and $(i-j)^{+}=\max\{0, i-j\}$. Therefore, it may be seen that if the cumulative demand process $\{N_t : t\geq 0\}$ forms a strictly unimodal distribution, the $\{IP_t : t\geq 0\}$ will converge in law to a uniform distribution.

Consider a cumulative probability distribution function $G(x)$ of which the probability density is defined with respect either to Lebesgue measure or counting measure. M is a mode for such a distribution function if $G(x)$ is a strictly convex function for $x\leq M$ but a strictly concave function for $x\geq M$. Moreover, such function G is strictly unimodal, and its probability density g is also a unimodal function having $g(y)>g(x)$ for $x<y\leq M$ and reversed for $x>y\geq M$.

Let $[x]$ be the largest integar less than or equal to x : that is, for $x\geq 0$, $0\leq x-[x]<1$.

Lemma 1.

Let the members $\{G_t\}$ of an index set of distributions have densities with respect to counting measure that are strictly unimodal for large enough t, with mode M_t. Also suppose that there exists a set $\{\alpha(t), \beta(t)\}$ of norming pairs (where $\beta(t)$ is an increasing positive-valued function), such that for all real Z,

$$\lim_{t\to\infty} G_t([\alpha(t)+Z\cdot\beta(t)])=G_*(Z). \qquad (1)$$

where G_* is a strictly unimodal distribution having a strictly unimodal density with respect to Lebesgue measure, with mode at $Z=0$.

Then, for $Z \gtrless 0$,

$[\alpha(t)+Z\cdot\beta(t)] \gtrless M_t$ for large enough t.

Proof.

Without loss of generality, consider the case $[\alpha(t)+Z\cdot\beta(t)]>M_t$ for $Z>0$.

If the conclusion were false, then there would exist a sequence $\{t_i\}$ such that as $t_i\to\infty$, $[\alpha(t_i)+Z\cdot\beta(t_i)]\leq M_{t_i}, \forall i$ and hence

$$[\alpha(t_i)+Z/2\cdot\beta(t_i)]\leq[\alpha(t_i)+Z\cdot\beta(t_i)]\leq M_{t_i}, \forall i,$$

so that

$$G_t([\alpha(t_i)])\cdot\left\{\frac{[\alpha(t_i)+Z\cdot\beta(t_i)]-[\alpha(t_i)+Z/2\cdot\beta(t_i)]}{[\alpha(t_i)+Z\cdot\beta(t_i)]-[\alpha(t_i)]}\right\} +G_t([\alpha(t_i)+Z\cdot\beta(t_i)])\cdot\left\{\frac{[\alpha(t_i)+Z/2\cdot\beta(t_i)]-[\alpha(t_i)]}{[\alpha(t_i)+Z\cdot\beta(t_i)]-[\alpha(t_i)]}\right\} \geq G_t([\alpha(t_i)+Z/2\cdot\beta(t_i)]) \qquad (2)$$

By definition,

$$R_1(t_i)=\frac{[\alpha(t_i)+Z\cdot\beta(t_i)]-[\alpha(t_i)+Z/2\cdot\beta(t_i)]}{[\alpha(t_i)+Z\cdot\beta(t_i)]-[\alpha(t_i)]} \geq\frac{\alpha(t_i)+Z\cdot\beta(t_i)-1-\alpha(t_i)-Z/2\cdot\beta(t_i)}{\alpha(t_i)+Z\cdot\beta(t_i)-\alpha(t_i)+1} =(1/2\cdot Z-1/\beta(t_i))/(Z+1/\beta(t_i)),$$

and

$$R_1(t_i)\leq\frac{\alpha(t_i)+Z\cdot\beta(t_i)-\alpha(t_i)-Z/2\cdot\beta(t_i)+1}{\alpha(t_i)+Z\cdot\beta(t_i)-1-\alpha(t_i)} =(1/2\cdot Z+1/\beta(t_i))/(Z-1/\beta(t_i)).$$

Therefore, $\lim_{t_i\to\infty} R_1(t_i)=1/2$, (3)

$$R_2(t_i)=\frac{[\alpha(t_i)+Z/2\cdot\beta(t_i)]-[\alpha(t_i)]}{[\alpha(t_i)+Z\cdot\beta(t_i)]-[\alpha(t_i)]} \geq\frac{\alpha(t_i)+Z/2\cdot\beta(t_i)-1-\alpha(t_i)}{\alpha(t_i)+Z\cdot\beta(t_i)-\alpha(t_i)+1}$$

$= (1/2 \cdot Z + 1/\beta(t_i)) / (Z + 1/\beta(t_i))$,

and

$$R_2(t_i) \leq \frac{\alpha(t_i) + Z/2 \cdot \beta(t_i) - \alpha(t_i) + 1}{\alpha(t_i) + Z \cdot \beta(t_i) - 1 - \alpha(t_i)}$$

$$= (1/2 \cdot Z + 1/\beta(t_i)) / (Z - 1/\beta(t_i)).$$

Also, $\lim_{t_i \to \infty} R_2(t_i) = \frac{1}{2}$ (4)

Now, taking limit on both sides of Eq. (2), from Eqs. (1), (3) and (4),

$1/2 (G_\circ(0) + G_\circ(Z)) \geq G_\circ(Z/2)$, and so

$G_\circ(0) + G_\circ(Z) \geq 2 \cdot G_\circ(Z/2)$

Therefore, $G_\circ(Z) - G_\circ(Z/2) \geq G_\circ(Z/2) - G_\circ(0)$

However, from the strictly concavity over $Z > 0$,

$G_0(Z) - G_0(Z/2) < G_0(Z/2) - G_0(0)$.

Hence, it is a contradiction.

Similarly, a contradiction can also so be shown for the case $[\alpha(t) + Z \cdot \beta(t)] < Mt$ for $Z < 0$.

Thus, the proof is complete.

Let $F(x)$ $(0 \leq x < \infty)$ be the distribution function of the demand interarrival process $\{X_n : n = 1, 2, \cdots\cdots\}$ and assume that $\mu = E(X_n) < \infty$, and $\sigma^2 = Var(X_n) < \infty$.

Prabhu [2] shows that if $t \to \infty$, the corresponding counting process $\{N_t\}$ has an asymptotically normal distribution with mean t/μ and variance $t\sigma^2/\mu^3$; that is,

$$\lim_{t \to \infty} Pr\left\{\frac{N_t - t/\mu}{\sqrt{t\sigma^2/\mu^3}} < x\right\} = 1/\sqrt{2\pi} \int_{-\infty}^{x} e^{-1/2y^2} dy. \quad (5)$$

It follows in view of Lemma 1 that this normality means the unimodality of the asymptotic distribution of $\{N_t\}$.

Corollary 1

When t is large enough, the eventual strict unimodality of the pre-asymptotic distribution of N_t is a sufficient condition that for $Z > 0$, the demand materializations $N_t = [t/\mu + Z\sqrt{t\sigma^2/\mu^3}]$ by time t are on the right side of the mode M_t.

Proof,

Suppose not. Then, there exists a subsequence $\{Z_j\}$ such that $[t/\mu + Z_j \cdot \sqrt{t\delta^2/\mu^3}] \leq M_t$, so that from Eq. (5).

$$\alpha_j = \int_0^{Z_j/2} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx - \int_{Z_j/2}^{Z_j} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx \leq 0,$$

which implies $\overline{\lim}\ \alpha_j \leq 0$. However, on the right side of the mode, $\lim \alpha_j > 0$, $\forall$ j. Therefore, it is a contradiction.

The above unimodality shall be applied to prove that the inventory position $\{IP_t\}$ in a $\langle Q, r \rangle$ system converges to a uniform distribution (see also Sivazlian [4]).

Theorem 1

Assume that the demand inter-arrival times X_j are iid random variables with a comon probability distribution F (with $F(0) = 0$). If for large enough t the distribution of the corresponding renewal counting process N_t is strictly unimodal, then, in the transactions reporting system with $\langle Q, r \rangle$ operating doctrine, the inventory position at time t IP_t converges in law to a uniform distribution.

Proof.

Let $\{n_i : i = 1, 2, \cdots\cdots\}$ be a sequence of nonnegative integers and let M_t denote the mode of the distribution of N_t.

Motivated by the relation

$$Pr\{IP_t = r + \ell\} = \sum_{i=0}^{\infty} Pr\{N_t = i \cdot Q + (Q - \ell)\}$$

for $\ell = 1, 2, \cdots\cdots, Q$,

following definitions are given:

Denoting $Pr\{N_t = i \cdot Q + (Q - \ell)\}$ by $P_t\{iQ + (Q - \ell)\}$,

$$P^{\ell}[n_1, n_2] \equiv \sum_{i=n_1}^{n_2-1} P_t\{i \cdot Q + (Q - \ell)\}$$

$$R_t[n_1, n_2] \equiv \sum_{i=n_1 Q}^{n_2 Q - 1} P_t\{i\} = \sum_{\ell=1}^{Q} \sum_{i=n_1}^{n_2-1} P_t\{iQ + (Q - \ell)\}$$

$$= \sum_{\ell=1}^{Q} P_t^{\ell}\{n_1, n_2\}$$

Let $[[x]]$ be the smallest integral multiple of the ordering amount Q greater than or equal to x:

if $(n-1)Q < x \leq nQ$, then $[[x]] = nQ$.

Let N be a set of nonnegative integers and let a finite subset E be a sequence of nonnegative integers $\{n_i : i=1, 2, \cdots, \mu\}$, for $\mu \geqq 1$, associated with the demand occurrences of N_t by time t.

Denote by $F\{L\}$ the probability distribution of N_t with respect to counting measure on the set $L \subset N$. Then, as $t \to \infty$, without loss of generality the finite subset E_μ can bs formed such that for arbitrary $\varepsilon > 0$.

(i) $F\{L\} < \varepsilon$, if $L = [0, n_i Q]$ or $L = [n_\mu Q, \infty]$

(ii) F is a strict convex function over $E_v \subset E_\mu$. where $1 \leqq v \leqq \mu$ and $n_v Q < M_t$.

(iii) F is a strict concave function over $\overline{E}_v \cap E_\mu$ with $n_{v+1} Q > M_t$,
where $\overline{E}_v$ is the complement of E_v and thus $n_{v+1} \in \overline{E}_v \cap E_\mu$.

Therefore, for ℓ, m=1, 2, ……, Q (but $\ell \neq$ m),

$$
\begin{aligned}
&|P_t^\ell[0,\infty] - P_t^m[0,\infty]| \leqq \{|P_t^\ell[0, n_1 Q] + P_t^m[0, n_1 Q]| \\
&\quad + \sum_{\alpha=1}^{v-1} |P_t^\ell[n_\alpha Q, n_{\alpha+1} Q] - P_t^m[n_\alpha Q, n_{\alpha+1} Q]| \\
&\quad + |P_t^\ell[n_v Q, n_{v+1} Q] + P_t^m[n_v Q, n_{v+1} Q]| \\
&\quad + \sum_{\beta=v+1}^{\mu-1} |P_t^\ell[n_\beta Q, n_{\beta+1} Q] - P_t^m[n_\beta Q, n_{\beta+1} Q]| \\
&\quad + |P_t^\ell[n_\mu Q, \infty] + P_t^m[n_\mu Q, \infty]|\} \\
&\leqq \{R_t[0, n_1 Q] \\
&\quad + \sum_{\alpha=1}^{v-1} |R_t[n_{\alpha+1} Q, n_{\alpha+1} Q + (n_{\alpha+1} -)] - R_t[n_\alpha Q - (n_{\alpha+1} - n_\alpha), n_\alpha Q]| \\
&\quad + R_t[n_v Q, n_{v+1} Q] \\
&\quad + \sum_{\beta=v+1}^{\mu-1} |R_t[n_{\beta+1} Q, n_{\beta+1} Q + (n_{\beta+1} - n_\beta)] - R_t[n_\beta Q - (n_{\beta+1} - n_\beta), n_\beta Q]| \\
&\quad + R_t[n_\mu Q, \infty]\},
\end{aligned} \tag{6}
$$

Let M_t be the mode of a standardized normal distribution $\Phi(Z)$, where $M_t = 0$ at $Z = 0$. Then consider a new finite sequence $\{Z_i : i=1, 2, \cdots, \mu\}$ of real values, which has the same sub-index set as that for E_μ, such that $Z_v < M_t$ and $Z_{v+1} > M_t$ for $1 \leqq v < \mu$. And also, in view of Lemma 1 and corollary 1, the relation $n_i Q = [[\mu(t) + Z_i \sigma(t)]]$ holds for all i's when $t \to \infty$, where $\{\mu(t), \sigma(t)\}$ is the norming pair satisfying $\lim_{t\to\infty} F_{N_t}\{[u(t) + Z \cdot \sigma(t)]\} = \lim_{t\to\infty} Pr\{N_t \leqq [\mu(t) + Z \cdot \sigma(t)]\} = \Phi(Z)$ for all real value Z.

Thus, it is clearly possible to form E_μ such that for $\varepsilon > 0$,

(i) $\Phi(Z_1) < \varepsilon$,

(ii) $|\Phi(Z_\mu) - \Phi(\infty)| < \varepsilon$, and

(iii) $|\Phi(Z_{v+1}) - \Phi(Z)| < \varepsilon$.

Moreover, for any positive integer multiplier λ $R_t[0, \lambda Q]$ can be made within ε of the corresponding $\Phi(Z_\lambda)$ uniformly; namely, $|R_t[0, \lambda Q] - \Phi(Z_\lambda)| < \varepsilon$, so that substituting $\Phi(\cdot)$ for $R_t[\cdot, \cdot]$ in the inequality (6) creates an error of at most $(\mu+1)\varepsilon$ in the partition of all real value Z.

Let $(Z_1, Z_2, \cdots, Z_v)$ and $(Z_{v+1}, Z_{v+2}, \cdots, Z_\mu)$ denote partitions of $[Z_1, Z_v]$ and $[Z_{v+1}, Z_\mu]$, respectively, such that intervals in each partition be equally spaced and following inequalties be satisfied ;

(i) $Z_2 - (Z_2 - Z_1)/Q \geqq Z_1$,

(ii) $Z_{v-1} + (Z_v - Z_{v-1})/Q \leqq Z_v$

(iii) $Z_{v+2} - (Z_{v+2} - Z_{v-1})/Q \geqq Z_{v+1}$

(iv) $Z_{\mu-1} + (Z_\mu - Z_{\mu-1})/Q \leqq Z_\mu$, and

for $\alpha = 1, 2, \cdots v-1$,

$$|\{\Phi(Z_{\alpha+1} + (Z_{\alpha+1} - Z_\alpha)/Q - \Phi(Z_{\alpha+1})\} - \{\Phi(Z_\alpha) - \Phi(Z_\alpha - (Z_{\alpha+1} - Z_\alpha)/Q)\}| < \varepsilon,$$

and for $\beta = v+1, v+2, \cdots, \mu-1$.

$$|\{\Phi(Z_{\beta+1} + (Z_{\beta+1} - Z_\beta)/Q) - \Phi(Z_{\beta+1})\} - \{\Phi(Z_\beta) - \Phi(Z_\beta - (Z_{\beta+1} - Z_\beta)/Q)\}| < \varepsilon.$$

According to such finite sequence constructions of $\{Z_i\}$ and $\{n_i\}$ for $i=1, 2, \cdots, \mu$, the inequality (6) is simplified as follows :

$$|P_t^\ell[0,\infty] - P_t^m[0,\infty]| < \{\varepsilon + (\mu-2)\varepsilon + \varepsilon + \varepsilon\} + (\mu+1)\varepsilon = 2(\mu+1)\varepsilon.$$

Since $2(\mu+1)$ is a finite number, $P_t^\ell[0,\infty]$ is asymptotically uniform when $t \to \infty$. This means that for $\ell=1, 2, \cdots\cdots, Q$,

$$\lim_{t\to\infty} \Pr\{IP_t = r+\ell\} = \lim_{t\to\infty} P_t^{\ell}[0,\infty] = \frac{1}{Q}$$

Therefore, the proof is complete.

From now on, we will discuss about the frame of proving that IP_t and $D_{(t+t+h)}$ are asymptotically independent for a nonnegative constant h as $t\to\infty$.

Under the assumption that n orders($n=0, 1, 2, \cdots\cdots$) have been placed by time t, $\ell+(n-1)Q+(Q-j)$ demands are presumed as the value of $N_t \equiv [t/\mu+Z_t\sqrt{t\sigma^2/\mu^3}]$ to locate the inventory position IP_t at a level $r+j$ with $IP_0=r+\ell$ ($j, \ell=1, 2, \cdots\cdots, Q$); that is,

$$\Pr\{IP_t=r+j\} = P_r\{N_t=\ell-j\}^+ + \sum_{n=1}^{\infty} P_r\{N_t = nQ+\ell-j\},$$

where $P_r\{N_t=\ell-j\}^+ = P_r\{N_t=\ell-j\}$, if $\ell \geq j$
$=0$, otherwise.

Let $[t/\mu+Z_i\sqrt{t\sigma^2/\mu^3}] = n_iQ+(\ell-j)$ and $[t/\mu+(Z_i+\delta)\sqrt{t\sigma^2/\mu^3}] = (n_i+m)Q+(\ell-j)$ ($i=1, 2, \cdots\cdots; b$), where m are non-negative integers. There also exist $0 \leq \xi_k < \delta$ for $k=0, 1, 2, \cdots\cdots, m-1$ and $\xi_m=\delta$ such that $[t/\mu+(Z_i+\xi_k)\sqrt{t\sigma^2/\mu^3}] = (n_i+k)Q+(\ell-j)$, so that the sum of probabilities (denoted by $\triangle_{ij}$) associated with $IP_t=r+j$ within the interval $[Z_i, Z_i+\delta]$ in Figure 1 is expressed as follows :

$$\triangle_{ij} = \sum_{k=0}^{m} P_r\{N_t=(n_i+k)Q+(\ell-j)\}$$
$$= \sum_{k=0}^{m} P_r\{N_t=[t/\mu+(Z_i+\xi_k)\sqrt{t\sigma^2/\mu^3}]\}$$

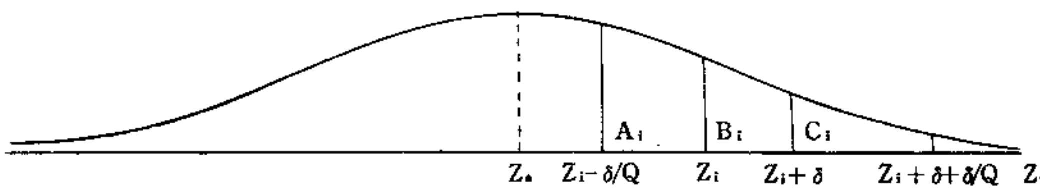

Fig 1. Normal Density Curve of Z_t

Assume that within the intervals of $[Z_i-\delta/Q, Z_i]$ and $[Z_i+\delta, Z_i+\delta+\delta/Q]$ there are also m different probabilities satisfying

$$\int_{Z_i-\delta}^{Z_i} f_{Z_t}(x)\,dx \cong \sum_{d=0}^{m} P_r\{N_t=n_iQ+(\ell-j)-d\} \text{ and}$$

$$\int_{Z_i+\delta}^{Z_i+\delta+\delta/Q} f_{Z_t}(x)\,dx \cong \sum_{d=0}^{m} P_r\{N_t=(n_i+m)Q+(\ell-j)+d\}, \text{ as } t\to\infty,$$

so the sums of probabilities in A_i, of each Q^{th} probabilities in β_i, and of probabilities in C_i can be compared with one another. Then, if we choose $\delta>0$ such that the ratio A_i/C_i absolutely converge to the value 1, it follows that

$$\frac{1}{Q}\int_{Z_i}^{Z_{i+1}} f_{Z_t}(x)\,dx \to \varnothing_{ij}, \forall i, \text{ as } t\to\infty.$$

This convergence will be the basic idea of proving the next theorem.

In Figure 1, the real line Z_t is broken down into a finite number $b+1$ of intervals $(-\infty, Z_1]$, $[Z_i, Z_{i+1}]$ ($i=1, 2, \cdots, a, a+1, \cdots; b-1$) and $[Z_b, \infty)$ such that $Z_a<0$, $Z_{a+1}>0$, and $Z_{i+1}-Z_i=\delta$, $\forall i$, where $\delta>0$ is chosen as follows ; for $\varepsilon>0$,

$$\int_{-\infty}^{Z_1} \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\,dx < \varepsilon, \quad \int_{Z_b}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\,dx < \varepsilon,$$

and on the right side of the mode M_t

$$\left|\frac{A_i}{C_i}-1\right| \leq \left|e^{\frac{1}{2}(2Z_i+\delta)(\delta+2\delta/Q)}-1\right| < \varepsilon$$

for $i=a, a+1, \cdots, b$, and on the left side of the mode

$$\left|\frac{A_i}{C_i}-1\right| \leq \left|e^{\frac{1}{2}(2Z_i+\delta)\cdot\delta}-1\right| < \varepsilon$$

for $i=1, 2, \cdots\cdots, a-1$. Whence $|A_i - C_i| < C_i \cdot \varepsilon$ and so $\sum_{i\in I} |A_i - C_i| < \sum_{i\in I} C_i \cdot \varepsilon \leq \varepsilon$, since $\sum_{i\in I} C_i \leq 1$ over $I = \{0, 1, 2, \cdots, a, a+1, \cdots\cdots, b, b+1\}$ with $Z_0 = -\infty$ and $Z_{b+1} = \infty$.

Moreover, if $Z_i - \delta/Q \geq M_t$, then

$$\overline{A_i} = \frac{1}{\sqrt{2\pi}} \cdot \frac{\delta}{Q} e^{-(Z_i - \delta/Q)^2/2} \geq A_i \equiv \int_{Z_i - \delta/Q}^{Z_i} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx \geq \frac{1}{\sqrt{2\pi}} \frac{\delta}{Q} e^{-Z_i^2/2} = \underline{A_i}.$$

$$\overline{B_i} = \frac{1}{\sqrt{2\pi}} \cdot \delta \cdot e^{-Z_i^2/2} \geq B_i \equiv \int_{Z_i}^{Z_i+\delta} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx \geq \frac{1}{\sqrt{2\pi}} \delta e^{-(Z_i+\delta)^2} = \underline{B_i}.$$

$$\underline{C_i} = \frac{1}{2\pi} \frac{\delta}{Q} e^{-(Z_i + \delta + \delta/Q)^2} \leq C_i \equiv \int_{Z_i+\delta}^{Z_i+\delta+\delta/Q} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx \leq \frac{1}{\sqrt{2\pi}} \frac{\delta}{Q} e^{-(Z_i+\delta)^2/2} = \overline{C_i}$$

and so $\underline{A_i} \geq \frac{1}{Q} \overline{B_i}$ and $\frac{1}{Q} \cdot \underline{B_i} \geq \overline{C_i}$. Thus, $A_i \geq \frac{1}{Q} \cdot B_i \geq C_i$, $\forall i$.

Theorem 2.

Let $\{X_n : n = 1, 2, \cdots\cdots\}$ be a renewal process with the identical distribution function $F(x)$ $(0 \leq x < \infty)$ and denote bx N_t the corresponding counting process.

If $F_{\frac{N_t - t/\mu}{\sqrt{t\sigma^2/\mu^3}}, D_{(t, t+h]}}(x, y)$ $(h \geq 0)$ asymptotically converges to a strictly unimodal distribution $\phi(x, y)$, then under a $\langle Q, r \rangle$ policy $F_{IP_t, D_{(t,t+h]}}(r+L, y)$ converges to $\frac{L}{Q} \phi(\cdot, y)$ as $t \to \infty$, where $L = \{1, 2, \cdots, Q\}$

Proof.

Let $Z_t = \frac{N_t - t/\mu}{\sqrt{t\sigma^2/\mu^3}}$. Assume that for $k \geq 0$ $\phi(x, y=k)$ is strictly unimodal, shown in Figure 2.

Let $z^j \in \Omega_j$ $(j=1, 2, \cdots\cdots, Q)$ form the demand materialization set of Z_t associated with

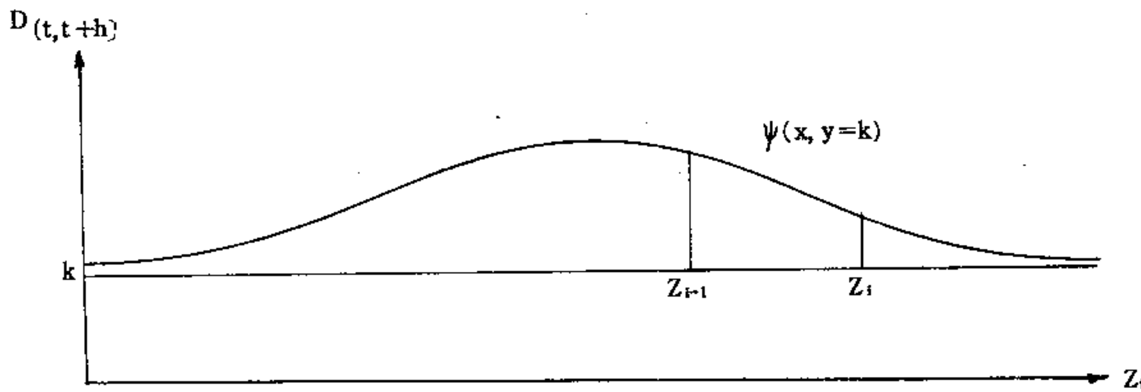

Fig 2. Asymptotically, strictly unimodal Density Curve of $F_{Z_t, D_{(t,t+h]}}(\cdot, y=k)$

$IP_t = r + j$, given $IP_0 = r + \ell$ $(\ell = 1, 2, \cdots\cdots, Q)$, such that under the assumption of n order placements by time t,

$$N_t = [t/\mu + z^j(n) \sqrt{t\sigma^2/\mu^3}] \equiv \ell + (n-1) \cdot Q + (Q-j) = nQ + (\ell - j).$$

where $Z^j(n_i) = Z_i^j + \xi_{m_i}^j$ with $0 \leq \xi_{m_i}^j \leq Z_i - Z_{i-1}$ $\forall i$ (as defined for Figure 1) corresponding to the relation $[t/\mu + (Z_{i-1}^j + \xi_{m_i}^j) \cdot \sqrt{t\sigma^2/\mu^3}] = [t/\mu + Z_{i-1}^j \sqrt{t\sigma^2/\mu^3}] + m_i Q$ for $m_i = 0, 1, 2, \cdots\cdots, [D^i/Q]$ with

$$D^i = [t/\mu + Z_i^j \sqrt{t\sigma^2/\mu^3}] - [t/\mu + Z_{i-1}^j \cdot \sqrt{t\sigma^2/\mu^3}].$$

Define followings: for $i=2, 3, \cdots\cdots, b$,

$$P^{j,k}_{i,m_i}(t)=P_r\{N_t=[t/\mu+(Z^j_{i-1}+\xi^j_{m_i})\cdot\sqrt{t\sigma^2/\mu^3}],\ D_{(t,t+h]}=k\},\ \text{and}$$

$$R^{j,k}_i(t)=\sum_{d=1}^{D_i}P_r\{N_t=[t/\mu+Z^j_{i-1}\sqrt{t\sigma^2/\mu^3}]+d,\ D_{(t,t+h]}=k\}.$$

By the assumptions,

$$\sum_{n=0}^{L(x)}P_r\{N_t=n,\ D_{(t,t+h]}=k\}\to F_{Z_t,\ D(t,t+h]}(y=k)$$

$$=P_r\{Z_t\leq x,\ D_{(t,t+h)}=k\}$$

as $t\to\infty$, where $L(x)\equiv[t/\mu+x\sqrt{t\sigma^2/\mu^3}]$

Therefore,

$R^{j,k}_i(t)\to\phi(Z^j_{i-1,i},\ y=k)$ as $t\to\infty$

where $Z^j_{i-1,i}=[Z_{i-1},Z_i]$

Moreover, under the $\langle Q,r\rangle$ model, Theorem 1 (see Sivazlian [4]) shows that

$$P_r\{IP_t=r+j\}=P_r\{N_t=\ell-j\}^+$$

$$+\sum_{n=1}^{\infty}P_r\{N_t=nQ+\ell-j\}\to\frac{1}{Q}\ \text{as}\ t\to\infty,$$

where $P_r\{N_t=\ell-j\}^+=P_r\{N_t=\ell-j\}$, if $\ell\geqq j$

$= 0$ otherwise.

It follows that

$$\sum_{m_i=0}^{[D^i/Q]}P^{j,k}_{i,m_i}(t)\cong\frac{1}{Q}\cdot R^{j,k}_i(t)$$

$$\to\frac{1}{Q}\cdot\phi(Z^j_{i-1,i}\ y=k).$$

Hence,

$$F_{IP_t,D_{(t,t+h]}}(r+L,y)=P_r\{IP_t\leq r+L,\ D_{(t,t+h)}\leq y\}\ (L=1,2,\cdots;Q)$$

$$=\sum_{j=1}^{L}\sum_{k=0}^{y}P_r\{IP_t=r+j,D_{(t,t+h]}=k\}$$

$$=\sum_{j=1}^{L}\sum_{k=0}^{y}(P_r\{N_t=\ell-j,\ D_{(t,t+h]}=k\}^+$$

$$+\sum_{n=1}^{\infty}P_r\{N_t=nQ+\ell-j,D_{(t,t+h]}=k\})$$

$$=\sum_{j=1}^{L}\sum_{k=0}^{y}(\sum_{n=0}^{\infty}P_r\{N_t=[t/\mu+Z^j_{n})\cdot\sqrt{t\sigma^2/\mu^3}],\ D_{(t,t+h]}=k\})$$

$$\cong\sum_{j=1}^{L}\sum_{k=0}^{y}\{\sum_{i=1}^{b}\sum_{m_i=0}^{[D^i/Q]}P^{j,k}_{i,m_i}(t)\}$$

$$\cong\sum_{j=1}^{L}\sum_{k=0}^{y}\{\sum_{i=1}^{b}\frac{1}{Q}\cdot\phi(Z^j_{i-1,i}\ y=k)\}$$

$$\cong\sum_{j=1}^{L}\sum_{k=0}^{y}\frac{1}{Q}\cdot\phi(\cdot,k)$$

$$=\frac{L}{Q}\cdot\phi(\cdot,y).$$

Thus, the proof is complete.

3. Conclusion

Through this study, a continuous-review inventory system $\langle Q,r\rangle$ was treated. Specifically, Theorem 2 shows that IP_t and $D_{(t,t+h]}$ are asymptotically mutually independent, when they form an asymptotic unimodal joint distribution. However, an extension to this work may be possible by analyzing the process $\{IP_t\}$ in view of a new renewal process at each reorder point.

The approach shown in this study ie expected easily applicable to investigate the periodic-review systems such as $\langle R,r,T\rangle$ systems.

Therefore, these results may be evaluated to lead to the analytical analyses of some inventory systems which have been treated under some restrictions on demand processes.

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